

\mathcal{D} -MODULES ON SMOOTH TORIC VARIETIESMIRCEA MUSTĂŢĂ, GREGORY G. SMITH, HARRISON TSAI,
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ABSTRACT. Let X be a smooth toric variety. Cox introduced the homogeneous coordinate ring S of X and its irrelevant ideal \mathfrak{b} . Let A denote the ring of differential operators on $\mathrm{Spec}(S)$. We show that the category of \mathcal{D} -modules on X is equivalent to a subcategory of graded A -modules modulo \mathfrak{b} -torsion. Additionally, we prove that the characteristic variety of a \mathcal{D} -module is a geometric quotient of an open subset of the characteristic variety of the associated A -module and that holonomic \mathcal{D} -modules correspond to holonomic A -modules.

1. INTRODUCTION

Let X be a smooth toric variety over a field k . Cox [C] introduced the homogeneous coordinate ring S of X and the irrelevant ideal \mathfrak{b} . The k -algebra S is a polynomial ring, with one variable for each one-dimensional cone in the fan Δ defining X , and has a natural grading by the class group $\mathrm{Cl}(X)$. The monomial ideal $\mathfrak{b} \subset S$ encodes the combinatorial structure of Δ . The following theorem of Cox [C] indicates the significance of the pair (S, \mathfrak{b}) . We write $\mathcal{O}\text{-Mod}$ for the category of quasi-coherent sheaves on X and $S\text{-GrMod}$ for the category of graded S -modules. A graded S -module F is called \mathfrak{b} -torsion if, for all $f \in F$, there exists $\ell > 0$ such that $\mathfrak{b}^\ell f = 0$. Let $\mathfrak{b}\text{-Tors}$ denote the full subcategory of \mathfrak{b} -torsion modules.

Theorem (Cox). 1. *The category $\mathcal{O}\text{-Mod}$ is equivalent to the quotient category $S\text{-GrMod}/\mathfrak{b}\text{-Tors}$.*
2. *The variety X is a geometric quotient of $\mathrm{Spec}(S) \setminus \mathrm{Var}(\mathfrak{b})$ by a suitable torus action.*

When $X = \mathbb{P}^n$, this is Serre's description of quasi-coherent sheaves on projective space and the classical construction of projective space.

The aim of this paper is to provide the \mathcal{D} -module version of this theorem — \mathcal{D} denotes the sheaf of differential operators on X . To state the analogue of the first part, we introduce the following notation. We

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write $\mathcal{D}\text{-Mod}$ for the category of left \mathcal{D} -modules on X . The ring of differential operators on $\text{Spec}(S)$ is the Weyl algebra A ; it also has a natural $\text{Cl}(X)$ grading. To each element $\bar{\mathbf{u}}$ in $\text{Cl}(X)^\vee = \text{Hom}_{\mathbb{Z}}(\text{Cl}(X), \mathbb{Z})$, we associate an “Euler” operator $\theta_{\bar{\mathbf{u}}} \in A$ [see (2.B) for the precise definition]. The full subcategory of graded left A -modules F satisfying $(\theta_{\bar{\mathbf{u}}} - \langle \bar{\mathbf{u}}, \bar{\mathbf{a}} \rangle) \cdot F_{\bar{\mathbf{a}}} = 0$ for all $\bar{\mathbf{a}} \in \text{Cl}(X)$ and all $\bar{\mathbf{u}} \in \text{Cl}(X)^\vee$ is denoted $A\text{-GrMod}_\theta$.

Theorem 1.1. *The quotient category $A\text{-GrMod}_\theta/\mathfrak{b}\text{-Tors}$ is equivalent to the category $\mathcal{D}\text{-Mod}$.*

The special case, when X is projective space, can be found in Borel $[B^+]$.

This categorical equivalence is given by two functors. The first takes an object F in $A\text{-GrMod}_\theta$ to the \mathcal{D} -module \tilde{F} whose sections over the affine open subset U_σ associated to $\sigma \in \Delta$ are $(F_{x\hat{\sigma}})_{\bar{\mathbf{0}}}$. The second maps a \mathcal{D} -module \mathcal{F} to $\Gamma_L(\mathcal{F}) = \bigoplus_{\bar{\mathbf{a}} \in \text{Cl}(X)} H^0(X, \mathcal{O}(\bar{\mathbf{a}}) \otimes \mathcal{F})$. In fact, if F is finitely generated then \tilde{F} is coherent and if \mathcal{F} is coherent then it is of the form \tilde{F} for some finitely generated graded A -module F . Our analysis of the $\Gamma_L(-)$ extends the work of Musson [M2] and Jones [J2] on rings of twisted differential operators on toric varieties.

Our second major result is

Theorem 1.2. *If $F \in A\text{-GrMod}_\theta$ is finitely generated, then the characteristic variety of \tilde{F} is a geometric quotient of a suitable open subset of the characteristic variety of F .*

Moreover, given a finitely generated $F \in A\text{-GrMod}_\theta$ which has no \mathfrak{b} -torsion, we show that the dimension of \tilde{F} is equal to the dimension of F minus the rank of $\text{Cl}(X)$. In particular, holonomic A -modules correspond to holonomic \mathcal{D} -modules.

The category of modules over the Weyl algebra is a well-studied algebraic object and we hope to study \mathcal{D} -modules on X by using these methods. In particular, effective algorithms have been developed for \mathcal{D} -modules on affine space; for example, see the work of Oaku [O], Walther [W], Saito-Sturmfels-Takayama [SST] and Oaku-Takayama [OT]. It would be interesting to use our results to extend these methods to smooth toric varieties.

Many of our results are valid for a simplicial toric variety if one replaces S and A with the subrings $\bigoplus_{\bar{\mathbf{b}} \in \text{Pic}(X)} S_{\bar{\mathbf{b}}}$ and $\bigoplus_{\bar{\mathbf{b}} \in \text{Pic}(X)} A_{\bar{\mathbf{b}}}$; see Cox [C]. This approach allows one to recover the general results of Musson [M2] and Jones [J2]. For simplicity, we present the smooth case and leave the possible generalizations to the reader.

The contents of this paper are as follows: The second section reviews the basics about toric varieties, the Weyl algebra and \mathcal{D} -modules. In

the third section, we determine the module associated with the sheaf $\mathcal{D} \otimes \mathcal{O}(\bar{\mathbf{b}})$. We introduce the $\mathrm{Cl}(X) \times \mathrm{Cl}(X)$ -graded A - A bimodule

$$D = \bigoplus_{\bar{\mathbf{b}} \in \mathrm{Cl}(X)} \frac{A(\bar{\mathbf{b}})}{A \cdot (\theta_{\bar{\mathbf{u}}} + \langle \bar{\mathbf{u}}, \bar{\mathbf{b}} \rangle : \bar{\mathbf{u}} \in \mathrm{Cl}(X)^\vee)}$$

and construct a morphism

$$\eta: D \longrightarrow \bigoplus_{(\bar{\mathbf{a}}, \bar{\mathbf{b}}) \in \mathrm{Cl}(X)^2} H^0(X, \mathcal{O}(\bar{\mathbf{a}}) \otimes \mathcal{D} \otimes \mathcal{O}(\bar{\mathbf{b}}))$$

We prove that η is an isomorphism in two steps. We first show that η induces an isomorphism on the associated sheaves. Secondly, we establish that $H_{\mathfrak{b}}^0(D) = H_{\mathfrak{b}}^1(D) = 0$. The first step is a local statement and can be reduced to results of Musson [M1]. We provide a direct proof using simplifications due to Jones [J1]. The fourth section contains the proof of Theorem 1.1. We establish this result for both left and right \mathcal{D} -modules. In general, there is an equivalence between these and we show how this is induced at the level of A -modules. In the last section, we prove Theorem 1.2 and related dimension results.

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2. BACKGROUND

We collect here a number of more or less standard definitions, results and notation. Throughout this paper, we work over an algebraically closed field k of characteristic zero.

Toric varieties. Let X be a smooth toric variety determined by the fan Δ in $N \cong \mathbb{Z}^n$. We write $\mathbf{v}_1, \dots, \mathbf{v}_d$ for the unique lattice vectors generating the one-dimensional cones in Δ and we assume that the \mathbf{v}_i span $N \otimes_{\mathbb{Z}} \mathbb{R}$. Each \mathbf{v}_i corresponds to an irreducible torus invariant Weil divisor in X . Since these divisors generate the torus invariant Weil divisors, we may identify the group of torus invariant Weil divisors with \mathbb{Z}^d . Let \mathbf{e}_i denote the standard basis for \mathbb{Z}^d and set $\mathbf{e} = \mathbf{e}_1 + \dots + \mathbf{e}_d$.

There is a short exact sequence

$$(2.A) \quad 0 \longrightarrow N^\vee \xrightarrow{\iota} \mathbb{Z}^d \xrightarrow{\overline{(\cdot)}} \mathrm{Cl}(X) \longrightarrow 0,$$

where $\iota(\mathbf{p}) = \langle \mathbf{p}, \mathbf{v}_1 \rangle \mathbf{e}_1 + \dots + \langle \mathbf{p}, \mathbf{v}_d \rangle \mathbf{e}_d$ and the second map $\mathbf{a} \mapsto \bar{\mathbf{a}}$ is the projection from Weil divisors to the divisor class group. Since X is smooth, the divisor class group $\mathrm{Cl}(X)$ is isomorphic to the Picard group $\mathrm{Pic}(X)$. In particular, the invertible sheaf (line bundle) associated to $\bar{\mathbf{a}} \in \mathrm{Cl}(X)$ is denoted $\mathcal{O}(\bar{\mathbf{a}})$.

Following [C], the homogeneous coordinate ring of X is the polynomial ring $S = k[x_1, \dots, x_d]$ with a $\text{Cl}(X)$ -grading induced by

$$\deg(x^{\mathbf{a}}) = \deg(x_1^{\mathbf{a}_1} \cdots x_d^{\mathbf{a}_d}) = \bar{\mathbf{a}} \in \text{Cl}(X).$$

For a cone $\sigma \in \Delta$, $\hat{\sigma}$ is the set $\{i : \mathbf{v}_i \notin \sigma\}$ and $x^{\hat{\sigma}} = \prod_{\mathbf{v}_i \notin \sigma} x_i$ is the associated monomial in S . The irrelevant ideal of X is the reduced monomial ideal $\mathfrak{b} = (x^{\hat{\sigma}} : \sigma \in \Delta)$.

Recall that each cone $\sigma \in \Delta$ corresponds to an open affine subset of X , $U_\sigma \cong \text{Spec}(S_{x^{\hat{\sigma}}})_{\mathfrak{b}}$. Every graded S -module F gives rise to a quasi-coherent sheaf on X , denoted by \tilde{F} , which corresponds to the module $(F_{x^{\hat{\sigma}}})_{\mathfrak{b}}$ over U_σ . If F is finitely generated over S , then \tilde{F} is a coherent \mathcal{O} -module; \mathcal{O} denotes the structure sheaf on X . Moreover, every quasi-coherent sheaf on X is of the form \tilde{F} for some graded S -module F and if the sheaf is coherent, then F can be taken finitely generated. For an S -module F , we have $\tilde{F} = 0$ if and only if $F = H_{\mathfrak{b}}^0(F)$; in other words F is \mathfrak{b} -torsion.

Weyl algebra. By definition, the d -th Weyl algebra is

$$A = \frac{k\{x_1, \dots, x_d, \partial_1, \dots, \partial_d\}}{\begin{pmatrix} x_i x_j - x_j x_i = 0 \\ \partial_i \partial_j - \partial_j \partial_i = 0 \\ \partial_i x_j - x_j \partial_i = \delta_{ij} \end{pmatrix}}.$$

The canonical ring morphism $S \hookrightarrow A$ provides A with the structure of left S -module. As in the case of S , A has a $\text{Cl}(X)$ -grading given by

$$\deg(x^{\mathbf{a}} \partial^{\mathbf{b}}) = \deg(x_1^{\mathbf{a}_1} \cdots x_d^{\mathbf{a}_d} \partial_1^{\mathbf{b}_1} \cdots \partial_d^{\mathbf{b}_d}) = \bar{\mathbf{a}} - \bar{\mathbf{b}} \in \text{Cl}(X),$$

and the $\bar{\mathbf{a}}$ -th graded component of A is denoted $A_{\bar{\mathbf{a}}}$. For each element $\bar{\mathbf{u}}$ of $\text{Cl}(X)^\vee = \text{Hom}_{\mathbb{Z}}(\text{Cl}(X), \mathbb{Z})$, we have an Euler operator

$$(2.B) \quad \theta_{\bar{\mathbf{u}}} = \langle \bar{\mathbf{u}}, \bar{\mathbf{e}}_1 \rangle \theta_1 + \cdots + \langle \bar{\mathbf{u}}, \bar{\mathbf{e}}_d \rangle \theta_d,$$

where $\theta_i = x_i \partial_i$. Notice that $\theta_{\bar{\mathbf{u}}}$ has degree zero.

The Weyl algebra A is isomorphic to the ring of differential operators on \mathbb{A}^d . The natural action of A on a polynomial $f \in S$ is $x_i \bullet f = x_i \cdot f$ and $\partial_i \bullet f = \frac{\partial f}{\partial x_i}$. Since S is also a subring of A , the symbol \bullet helps distinguish this action from the product $\cdot : A \times A \rightarrow A$. The ring of differential operators on $S_{x^{\mathbf{a}}} = S[x^{-\mathbf{a}}]$ is denoted $A_{x^{\mathbf{a}}}$ and is equal to the localization $A[x^{-\mathbf{a}}]$.

\mathcal{D} -modules. The sheaf of (algebraic) differential operators on X is denoted \mathcal{D} . On an affine open subset $U \subseteq X$, $H^0(U, \mathcal{D}) = \bigcup_{i \geq 0} \mathcal{D}^i(U)$ where $\mathcal{D}^0(U) = H^0(U, \mathcal{O})$ and

$$\mathcal{D}^i(U) := \left\{ s \in \text{End}_k(H^0(U, \mathcal{O})) : \begin{array}{l} fs - sf \in \mathcal{D}^{i-1}(U) \\ \text{for all } f \in H^0(U, \mathcal{O}) \end{array} \right\}.$$

A \mathcal{D} -module is a sheaf \mathcal{F} on X which is quasi-coherent as an \mathcal{O} -module and has a structure of module over \mathcal{D} . A \mathcal{D} -module is coherent if it is locally finitely generated over \mathcal{D} . We write $\mathcal{D}\text{-Mod}$ and $\text{Mod-}\mathcal{D}$ for the categories of left and right \mathcal{D} -modules respectively. The full subcategories of coherent left and right \mathcal{D} -modules are denoted $\mathcal{D}\text{-Coh}$ and $\text{Coh-}\mathcal{D}$.

3. SHEAVES OF DIFFERENTIAL OPERATORS

The goal of this section is to describe the left S -modules corresponding to twists of the sheaf of differential operators. Recall that, for a graded A -module F and $\bar{\mathbf{b}} \in \text{Cl}(X)$, $F(\bar{\mathbf{b}})$ is the shift of F by $\bar{\mathbf{b}}$: $F(\bar{\mathbf{b}})_{\bar{\mathbf{a}}} = F_{\bar{\mathbf{b}}+\bar{\mathbf{a}}}$. We define the graded left A -modules

$$D_L(\bar{\mathbf{b}}) = \frac{A(\bar{\mathbf{b}})}{A \cdot (\theta_{\bar{\mathbf{u}}} + \langle \bar{\mathbf{u}}, \bar{\mathbf{b}} \rangle : \bar{\mathbf{u}} \in \text{Cl}(X)^\vee)}$$

and

$$D = \bigoplus_{\bar{\mathbf{b}} \in \text{Cl}(X)} D_L(\bar{\mathbf{b}}).$$

Notice that D has a $\text{Cl}(X) \times \text{Cl}(X)$ -grading where $D_{(\bar{\mathbf{a}}, \bar{\mathbf{b}})} = D_L(\bar{\mathbf{b}})_{\bar{\mathbf{a}}}$.

Lemma 3.1. *The module D is a graded A - A bimodule.*

Proof. Multiplication in the ring A yields the right action of A on D :

$$\begin{array}{ccc} A(\bar{\mathbf{b}})_{\bar{\mathbf{a}}} \otimes_k A_{\bar{\mathbf{b}'}} & \longrightarrow & A(\bar{\mathbf{b}} + \bar{\mathbf{b}}')_{\bar{\mathbf{a}}} \\ \downarrow & & \downarrow \\ D_L(\bar{\mathbf{b}})_{\bar{\mathbf{a}}} \otimes_k A_{\bar{\mathbf{b}'}} & \longrightarrow & D_L(\bar{\mathbf{b}} + \bar{\mathbf{b}}')_{\bar{\mathbf{a}}}. \end{array}$$

To see that the induced map is well-defined, observe that, for all elements $f \in A_{\bar{\mathbf{b}'}}$ and $\bar{\mathbf{u}} \in \text{Cl}(X)^\vee$, we have

$$\begin{aligned} (\theta_{\bar{\mathbf{u}}} + \langle \bar{\mathbf{u}}, \bar{\mathbf{b}} \rangle) \cdot f &= f \cdot (\theta_{\bar{\mathbf{u}}} + \langle \bar{\mathbf{u}}, \bar{\mathbf{b}} \rangle) + \langle \bar{\mathbf{u}}, \bar{\mathbf{b}}' \rangle \cdot f \\ &= f \cdot (\theta_{\bar{\mathbf{u}}} + \langle \bar{\mathbf{u}}, \bar{\mathbf{b}} + \bar{\mathbf{b}}' \rangle). \end{aligned}$$

This action is clearly compatible with the left A -module structure. It follows that D is an A - A bimodule. Since $f \in A_{\bar{\mathbf{a}'}}$ and $g \in A_{\bar{\mathbf{b}'}}$ implies $f \cdot D_{(\bar{\mathbf{a}}, \bar{\mathbf{b}})} \cdot g \subseteq D_{(\bar{\mathbf{a}}+\bar{\mathbf{a}'}, \bar{\mathbf{b}}+\bar{\mathbf{b}}')}$, D is bigraded. In other words, if we let A° denote the opposite algebra, then $A \otimes_k A^\circ$ is a $\text{Cl}(X)^2$ -graded ring and D is a graded module over $A \otimes_k A^\circ$. \square

Analogously, we define right A -modules:

$$D_R(\bar{\mathbf{a}}) = \frac{A(\bar{\mathbf{a}})}{(\theta_{\bar{\mathbf{u}}} - \langle \bar{\mathbf{u}}, \bar{\mathbf{a}} \rangle : \bar{\mathbf{u}} \in \text{Cl}(X)^\vee) \cdot A}$$

and

$$D' = \bigoplus_{\bar{\mathbf{a}} \in \text{Cl}(X)} D_R(\bar{\mathbf{a}}).$$

Again, D' is a $\text{Cl}(X) \times \text{Cl}(X)$ -graded A - A bimodule where the multiplication on the left is induced by the multiplication in the Weyl algebra. In fact, we obtain the same module.

Lemma 3.2. *There is a canonical identification $D = D'$ which respects the graded bimodule structure.*

Proof. Since we have

$$D_{(\bar{\mathbf{a}}, \bar{\mathbf{b}})} = \frac{A_{\bar{\mathbf{a}} + \bar{\mathbf{b}}}}{A_{\bar{\mathbf{a}} + \bar{\mathbf{b}}} \cdot (\theta_{\bar{\mathbf{u}}} + \langle \bar{\mathbf{u}}, \bar{\mathbf{b}} \rangle : \bar{\mathbf{u}} \in \text{Cl}(X)^\vee)}$$

and

$$D'_{(\bar{\mathbf{a}}, \bar{\mathbf{b}})} = \frac{A_{\bar{\mathbf{a}} + \bar{\mathbf{b}}}}{(\theta_{\bar{\mathbf{u}}} - \langle \bar{\mathbf{u}}, \bar{\mathbf{a}} \rangle : \bar{\mathbf{u}} \in \text{Cl}(X)^\vee) \cdot A_{\bar{\mathbf{a}} + \bar{\mathbf{b}}}},$$

it is enough to show that $A_{\bar{\mathbf{a}} + \bar{\mathbf{b}}} \cdot (\theta_{\bar{\mathbf{u}}} + \langle \bar{\mathbf{u}}, \bar{\mathbf{b}} \rangle) = (\theta_{\bar{\mathbf{u}}} - \langle \bar{\mathbf{u}}, \bar{\mathbf{a}} \rangle) \cdot A_{\bar{\mathbf{a}} + \bar{\mathbf{b}}}$. However, for every $f \in A_{\bar{\mathbf{a}} + \bar{\mathbf{b}}}$, we have

$$\begin{aligned} f \cdot (\theta_{\bar{\mathbf{u}}} + \langle \bar{\mathbf{u}}, \bar{\mathbf{b}} \rangle) &= (\theta_{\bar{\mathbf{u}}} + \langle \bar{\mathbf{u}}, \bar{\mathbf{b}} \rangle) \cdot f - \langle \bar{\mathbf{u}}, \bar{\mathbf{a}} + \bar{\mathbf{b}} \rangle \cdot f \\ &= (\theta_{\bar{\mathbf{u}}} - \langle \bar{\mathbf{u}}, \bar{\mathbf{a}} \rangle) \cdot f, \end{aligned}$$

which establishes the lemma. \square

Now, the cohomology of \mathcal{D} has an A - A bimodule structure.

Lemma 3.3. *The direct sum*

$$\bigoplus_{(\bar{\mathbf{a}}, \bar{\mathbf{b}}) \in \text{Cl}(X)^2} H^0(X, \mathcal{O}(\bar{\mathbf{a}}) \otimes \mathcal{D} \otimes \mathcal{O}(\bar{\mathbf{b}}))$$

is an A - A bimodule.

Proof. It suffices to give k -linear maps:

$$\begin{aligned} (3.A) \quad \mu: A_{\bar{\mathbf{a}}'} \otimes_k H^0(X, \mathcal{O}(\bar{\mathbf{a}}) \otimes \mathcal{D} \otimes \mathcal{O}(\bar{\mathbf{b}})) \otimes_k A_{\bar{\mathbf{b}}'} \\ \longrightarrow H^0(X, \mathcal{O}(\bar{\mathbf{a}} + \bar{\mathbf{a}}') \otimes \mathcal{D} \otimes \mathcal{O}(\bar{\mathbf{b}} + \bar{\mathbf{b}}')). \end{aligned}$$

Locally, a section $s \in H^0(U_\sigma, \mathcal{O}(\bar{\mathbf{a}}) \otimes \mathcal{D} \otimes \mathcal{O}(\bar{\mathbf{b}}))$ can be identified with an element of $\text{Hom}_k((S_{x^{\hat{\sigma}}})_{-\bar{\mathbf{b}}}, (S_{x^{\hat{\sigma}}})_{\bar{\mathbf{a}}})$, where σ is a cone in Δ . Moreover, the action of $f \in A_{\bar{\mathbf{a}}}$ on S descends to action on $S_{x^{\hat{\sigma}}}$ which increases degrees by $\bar{\mathbf{a}}$. Thus, we may define $(\mu|_{U_\sigma})(f \otimes s \otimes g) = f \circ s \circ g$. One verifies that $\mu|_{U_\sigma}$ maps into $H^0(U_\sigma, \mathcal{O}(\bar{\mathbf{a}} + \bar{\mathbf{a}}') \otimes \mathcal{D} \otimes \mathcal{O}(\bar{\mathbf{b}} + \bar{\mathbf{b}}'))$ and that these local definitions glue to give the required map. \square

We next construct a morphism of graded A - A bimodules

$$(3.B) \quad \eta: D \longrightarrow \bigoplus_{(\bar{\mathbf{a}}, \bar{\mathbf{b}})} H^0(X, \mathcal{O}(\bar{\mathbf{a}}) \otimes \mathcal{D} \otimes \mathcal{O}(\bar{\mathbf{b}})) .$$

Since η is a graded k -linear morphism, it is enough to define $\eta(f)$ for $f \in D_{(\bar{\mathbf{a}}, \bar{\mathbf{b}})}$. For $f \in D_{(\bar{\mathbf{a}}, \bar{\mathbf{b}})}$, let $\eta(f)$ be the section of $H^0(X, \mathcal{O}(\bar{\mathbf{a}}) \otimes \mathcal{D} \otimes \mathcal{O}(\bar{\mathbf{b}}))$ whose restriction over each U_σ corresponds to the map induced by the action of $f: (S_{x^{\hat{\sigma}}})_{-\bar{\mathbf{b}}} \longrightarrow (S_{x^{\hat{\sigma}}})_{\bar{\mathbf{a}}}$. To see that $\eta(f)$ is well-defined, consider $f \in A_{\bar{\mathbf{a}}+\bar{\mathbf{b}}} \cdot (\theta_{\bar{\mathbf{u}}} + \langle \bar{\mathbf{u}}, \bar{\mathbf{b}} \rangle)$. It follows that, for $g \in (S_{x^{\hat{\sigma}}})_{-\bar{\mathbf{b}}}$, we have

$$(\theta_{\bar{\mathbf{u}}} + \langle \bar{\mathbf{u}}, \bar{\mathbf{b}} \rangle) \bullet g = -\langle \bar{\mathbf{u}}, \bar{\mathbf{b}} \rangle \cdot g + \langle \bar{\mathbf{u}}, \bar{\mathbf{b}} \rangle \cdot g = 0$$

and therefore $\eta(f) = 0$. It is clear that η is a morphism of graded A - A bimodules. The main result of this section is the following.

Theorem 3.4. *The morphism η [see equation (3.B)] is an isomorphism of graded A - A bimodules.*

Before proving Theorem 3.4, we collect some local results. We first consider a local version of η . By composing η with the restriction to U_σ where $\sigma \in \Delta$, we obtain a morphism of left S -modules

$$D \longrightarrow \bigoplus_{(\bar{\mathbf{a}}, \bar{\mathbf{b}}) \in \text{Cl}(X)^2} H^0(U_\sigma, \mathcal{O}(\bar{\mathbf{a}}) \otimes \mathcal{D} \otimes \mathcal{O}(\bar{\mathbf{b}})) ,$$

which induces a morphism

$$\eta^\sigma: D_{x^{\hat{\sigma}}} \longrightarrow \bigoplus_{(\bar{\mathbf{a}}, \bar{\mathbf{b}}) \in \text{Cl}(X)^2} H^0(U_\sigma, \mathcal{O}(\bar{\mathbf{a}}) \otimes \mathcal{D} \otimes \mathcal{O}(\bar{\mathbf{b}})) .$$

Taking the degree zero component yields the ring morphism

$$(3.C) \quad \bar{\varphi}_\sigma = \eta_{(\bar{\mathbf{0}}, \bar{\mathbf{0}})}^\sigma: \frac{(A_{x^{\hat{\sigma}}})_{\bar{\mathbf{0}}}}{(A_{x^{\hat{\sigma}}})_{\bar{\mathbf{0}}} \cdot (\theta_{\bar{\mathbf{u}}} : \bar{\mathbf{u}} \in \text{Cl}(X)^\vee)} \longrightarrow H^0(U_\sigma, \mathcal{D}) .$$

Understanding $\bar{\varphi}_\sigma$ is an important ingredient in establishing Theorem 3.4. We begin by studying the map $\varphi_\sigma: (A_{x^{\hat{\sigma}}})_{\bar{\mathbf{0}}} \longrightarrow H^0(U_\sigma, \mathcal{D})$ which induces $\bar{\varphi}_\sigma$.

By definition, $H^0(U_\sigma, \mathcal{D})$ is the ring of differential operators on the ring $(S_{x^{\hat{\sigma}}})_{\bar{\mathbf{0}}}$. However, the inclusion $\iota_\sigma: \sigma^\vee \cap N^\vee \hookrightarrow \mathbb{Z}^d$ induces a ring isomorphism (denoted by the same name) $\iota_\sigma: k[\sigma^\vee \cap N^\vee] \xrightarrow{\cong} (S_{x^{\hat{\sigma}}})_{\bar{\mathbf{0}}}$. To see this, observe that $x^{\iota(\mathbf{p})} \in S_{x^{\hat{\sigma}}}$ if and only if $\langle \mathbf{p}, \mathbf{v}_i \rangle \geq 0$, for all $\mathbf{v}_i \in \sigma$, which is equivalent to $\mathbf{p} \in \sigma^\vee$. It follows that the isomorphism ι_σ induces an isomorphism, called ψ_σ , from the differential operators on $(S_{x^{\hat{\sigma}}})_{\bar{\mathbf{0}}}$ to the differential operators R_σ on $k[\sigma^\vee \cap N^\vee]$. More explicitly,

we have $\psi_\sigma(f) = \iota_\sigma^{-1} \circ f \circ \iota_\sigma$. We will actually focus on the morphism $\psi_\sigma \circ \varphi_\sigma : (A_{x^{\hat{\sigma}}})_{\bar{0}} \longrightarrow R_\sigma$.

Following [M1] and [J1], we decompose $(A_{x^{\hat{\sigma}}})_{\bar{0}}$ and R_σ under the appropriate torus actions and express $\psi_\sigma \circ \varphi_\sigma$ in terms of these decompositions. To be more concrete, we identify N^\vee with \mathbb{Z}^n by fixing a basis. Let $\varepsilon_1, \dots, \varepsilon_n$ denote the standard basis of \mathbb{Z}^n and let $\varepsilon = \varepsilon_1 + \dots + \varepsilon_n$. We continue to call the natural embedding $\iota : N^\vee = \mathbb{Z}^n \longrightarrow \mathbb{Z}^d$.

The torus $(k^*)^d$ acts on $k[x_1^{\pm 1}, \dots, x_d^{\pm 1}]$ by $\lambda * (x^{\mathbf{a}}) = \lambda^{\mathbf{a}} x^{\mathbf{a}}$ which produces an action on $A_{x^{\mathbf{e}}}$ given by $\lambda * (x^{\mathbf{a}} \partial^{\mathbf{b}}) = \lambda^{\mathbf{a}-\mathbf{b}} x^{\mathbf{a}} \partial^{\mathbf{b}}$ for $\lambda \in (k^*)^d$. The corresponding eigenspace decomposition is $A_{x^{\mathbf{e}}} = \bigoplus_{\mathbf{a} \in \mathbb{Z}^d} x^{\mathbf{a}} \cdot W$, where W is the polynomial ring $k[\theta_1, \dots, \theta_d]$. Taking the degree zero part, we have $(A_{x^{\mathbf{e}}})_{\bar{0}} = \bigoplus_{\mathbf{p} \in \mathbb{Z}^n} x^{\iota(\mathbf{p})} \cdot W$. Since $(A_{x^{\hat{\sigma}}})_{\bar{0}}$ is invariant under the action of $(k^*)^d$, the decomposition of $(A_{x^{\hat{\sigma}}})_{\bar{0}}$ yields

$$(A_{x^{\hat{\sigma}}})_{\bar{0}} = \bigoplus_{\mathbf{p} \in \mathbb{Z}^n} x^{\iota(\mathbf{p})} \cdot J(\mathbf{p}),$$

where $J(\mathbf{p})$ is an ideal in W . To describe $J(\mathbf{p})$, recall that $A_{x^{\hat{\sigma}}}$ is the ring of differential operators on $S_{x^{\hat{\sigma}}}$. Thus, if $g \in W$, then $x^{\iota(\mathbf{p})}g$ belongs to $J(\mathbf{p})$ if and only if $(x^{\iota(\mathbf{p})}g) \bullet S_{x^{\hat{\sigma}}} \subseteq S_{x^{\hat{\sigma}}}$. Equivalently, for every $x^{\mathbf{a}}$ satisfying $\mathbf{a}_i \geq 0$ when $\mathbf{v}_i \in \sigma$, we have $g(\mathbf{a})x^{\iota(\mathbf{p})+\mathbf{a}} \in S_{x^{\hat{\sigma}}}$. We conclude that $J(\mathbf{p})$ is the ideal of polynomials vanishing on

$$Z(\mathbf{p}) = \left\{ \mathbf{a} \in \mathbb{Z}^d : \begin{array}{l} \mathbf{a}_i \geq 0 \text{ if } \mathbf{v}_i \in \sigma \text{ and} \\ \iota(\mathbf{p})_j + \mathbf{a}_j < 0 \text{ for some } \mathbf{v}_j \in \sigma \end{array} \right\}.$$

Analogously, the affine space \mathbb{A}_k^n has as its associated Weyl algebra,

$$A' = \frac{k\{y_1, \dots, y_n, \bar{\partial}_1, \dots, \bar{\partial}_n\}}{\left(\begin{array}{l} y_i y_j - y_j y_i = 0 \\ \bar{\partial}_i \bar{\partial}_j - \bar{\partial}_j \bar{\partial}_i = 0 \\ \bar{\partial}_i y_j - y_j \bar{\partial}_i = \delta_{ij} \end{array} \right)}$$

and the torus $(k^*)^n$ acts on the Laurent ring $k[y_1^{\pm 1}, \dots, y_n^{\pm 1}] = k[N^\vee]$ inducing an action on $A'_{y^{\varepsilon}}$. The corresponding eigenspace decomposition is $A'_{y^{\varepsilon}} = \bigoplus_{\mathbf{p} \in \mathbb{Z}^n} y^{\mathbf{p}} \cdot W'$, where W' is the polynomial ring $k[\vartheta_1, \dots, \vartheta_n]$ and $\vartheta_i = y_i \bar{\partial}_i$. The inclusion $R_\sigma \hookrightarrow A'_{y^{\varepsilon}}$ identifies R_σ with

$$\{f \in A'_{y^{\varepsilon}} : f \bullet (k[\sigma^\vee \cap N^\vee]) \subseteq k[\sigma^\vee \cap N^\vee]\}.$$

Since this condition is torus invariant, R_σ is also torus invariant and we obtain $R_\sigma = \bigoplus_{\mathbf{p} \in \mathbb{Z}^n} y^{\mathbf{p}} \cdot I(\mathbf{p})$, where

$$I(\mathbf{p}) = \{f \in W' : (y^{\mathbf{p}} f) \bullet (k[\sigma^\vee \cap N^\vee]) \subseteq k[\sigma^\vee \cap N^\vee]\}.$$

Identifying W' with the coordinate ring of \mathbb{A}^n , we have $f \bullet y^{\mathbf{q}} = f(\mathbf{q})y^{\mathbf{q}}$ for every $\mathbf{q} \in N^\vee$. Hence, $I(\mathbf{p})$ is the ideal of polynomials vanishing on

$$Y(\mathbf{p}) = \{\mathbf{q} \in \sigma^\vee \cap N^\vee : \mathbf{q} + \mathbf{p} \notin \sigma^\vee \cap N^\vee\}.$$

Finally, we define a map $\rho: W \longrightarrow W'$. The inclusion $\iota: \mathbb{Z}^n \rightarrow \mathbb{Z}^d$ induces, by tensoring with k , a linear embedding (denoted by the same name) $\iota: \mathbb{A}^n \rightarrow \mathbb{A}^d$ of the corresponding affine spaces. Identifying W and W' with the coordinate rings of \mathbb{A}^d and \mathbb{A}^n respectively, we obtain the ring homomorphism $\rho = \iota^*: W \rightarrow W'$. Clearly, ρ is surjective and $\text{Ker}(\rho) = (\theta_{\bar{\mathbf{u}}} : \bar{\mathbf{u}} \in \text{Cl}(X)^\vee)$.

With this notation, we have

Lemma 3.5. *The ring homomorphism $\psi_\sigma \circ \varphi_\sigma : (A_{x^\sigma})_{\bar{\mathbf{0}}} \longrightarrow R_\sigma$ is given by $x^{\iota(\mathbf{p})} \cdot g \mapsto y^{\mathbf{p}} \cdot \rho(g)$ where $\mathbf{p} \in \mathbb{Z}^n$ and $g \in J(\mathbf{p})$.*

Proof. It suffices to show that $(\psi_\sigma \circ \varphi_\sigma)(x^{\iota(\mathbf{p})}g)$ and $y^{\mathbf{p}}\rho(g)$ have the same action on $y^{\mathbf{q}} \in k[\sigma^\vee \cap N^\vee]$. On one hand, we have

$$\begin{aligned} (\psi_\sigma \circ \varphi_\sigma)(x^{\iota(\mathbf{p})}g) \bullet y^{\mathbf{q}} &= \iota^{-1}(\varphi_\sigma(x^{\iota(\mathbf{p})}g \bullet \iota(y^{\mathbf{q}})) \\ &= \iota^{-1}(\varphi_\sigma(x^{\iota(\mathbf{p})}g \bullet x^{\iota(\mathbf{q})}) \\ &= \iota^{-1}(g(\iota(\mathbf{q}))x^{\iota(\mathbf{p}+\mathbf{q})}) \\ &= g(\iota(\mathbf{q}))y^{\mathbf{p}+\mathbf{q}}. \end{aligned}$$

On the other hand, we also have

$$(y^{\mathbf{p}}\rho(g)) \bullet y^{\mathbf{q}} = (\rho(g)(\mathbf{q}))y^{\mathbf{p}+\mathbf{q}} = g(\iota(\mathbf{q}))y^{\mathbf{p}+\mathbf{q}},$$

which establishes the claim. \square

Before returning our attention to $\bar{\varphi}_\sigma$, we need one more lemma.

Lemma 3.6. *If the elements θ_i and θ_j in $W = k[\theta_1, \dots, \theta_d]$ are distinct and correspond to rays \mathbf{v}_i and \mathbf{v}_j in the same cone σ , then, for every pair of integers m_i and m_j , the elements $\rho(\theta_i) + m_i$ and $\rho(\theta_j) + m_j$ are linearly independent over the field k .*

Proof. Suppose otherwise: for some $c \in k$, we have $\rho(\theta_i) + m_i = c \cdot (\rho(\theta_j) + m_j)$. It follows that $\rho(\theta_i) = c \cdot \rho(\theta_j)$ and hence

$$\sum_{\ell=1}^n \langle \varepsilon_\ell, \mathbf{v}_i \rangle \vartheta_\ell = c \cdot \left(\sum_{\ell=1}^n \langle \varepsilon_\ell, \mathbf{v}_j \rangle \vartheta_\ell \right).$$

We deduce that $\langle \varepsilon_\ell, \mathbf{v}_i \rangle = \langle \varepsilon_\ell, c \cdot \mathbf{v}_j \rangle$ for all ℓ and thus $\mathbf{v}_i = c \cdot \mathbf{v}_j$. However, the \mathbf{v}_i in any cone σ are linearly independent. \square

We are now in a position to understand $\bar{\varphi}_\sigma$. In particular, we obtain the following proposition which is a special case of Musson's results on rings of differential operators; see [M1].

Proposition 3.7 (Musson). *For every $\sigma \in \Delta$, the map $\bar{\varphi}_\sigma$ [see equation (3.C)] is an isomorphism of rings.*

Proof. The fact that $\bar{\varphi}_\sigma$ is a ring homomorphism follows directly from the definition. Thus, the assertion reduces to showing that $\psi_\sigma \circ \varphi_\sigma$ is surjective and to describe its kernel. To achieve this, we determine the Zariski closures of $Y(\mathbf{p})$ and $Z(\mathbf{p})$. Since the Zariski closure of the set of integer points inside a rational polyhedral cone is the linear space spanned by that cone, it is easy to check that

$$(3.D) \quad \begin{cases} \overline{Z(\mathbf{p})} &= \bigcup_{(i,m) \in \Lambda'} \{\mathbf{b} \in k^d : b_i = m\}, \\ \overline{Y(\mathbf{p})} &= \bigcup_{(i,m) \in \Lambda'} \{\mathbf{q} \in k^n : \iota(\mathbf{q})_i = m\} = \iota^{-1}(\overline{Z(\mathbf{p})}), \end{cases}$$

where $\Lambda' = \{(i, m) : \mathbf{v}_i \in \sigma \text{ and } 0 \leq m \leq -\iota(\mathbf{p})_i\}$. We claim that $\rho(J(\mathbf{p})) = I(\mathbf{p})$. Indeed, equation (3.D) implies that $I(\mathbf{p}) = \sqrt{\rho(J(\mathbf{p}))}$, so it is enough to check that $\rho(J(\mathbf{p}))$ is a radical ideal. However, $J(\mathbf{p})$ is the principal ideal generated by

$$(3.E) \quad h_{\mathbf{p}} := \prod_{(i,m) \in \Lambda'} (\theta_i - m).$$

From the equation (3.E) and Lemma 3.6, we see that $\rho(J(\mathbf{p}))$ is reduced and hence $I(\mathbf{p}) = \rho(J(\mathbf{p}))$. Applying Lemma 3.5, it follows that $\psi_\sigma \circ \varphi_\sigma$ and $\bar{\varphi}_\sigma$ are surjective. To prove that $\psi_\sigma \circ \bar{\varphi}_\sigma$ is injective, recall that $\text{Ker}(\rho) = (\theta_{\bar{\mathbf{u}}} : \bar{\mathbf{u}} \in \text{Cl}(X)^\vee)$. Thus, it is enough to show that, for every $\mathbf{p} \in \mathbb{Z}^n$, we have $J(\mathbf{p}) \cap \text{Ker}(\rho) = J(\mathbf{p}) \cdot \text{Ker}(\rho)$. To see this, observe that $f \in J(\mathbf{p}) \cap \text{Ker}(\rho)$ implies $f = h_{\mathbf{p}} f_1$. Therefore, it suffices to notice that $\rho(h_{\mathbf{p}}) \neq 0$, which follows from Lemma 3.6. \square

We now prove the main result in this section.

Proof of Theorem 3.4. We must show that, for every $\bar{\mathbf{b}} \in \text{Cl}(X)$, the homomorphism of graded left A -modules

$$\eta_{(*, \bar{\mathbf{b}})} : D_L(\bar{\mathbf{b}}) \longrightarrow \bigoplus_{\bar{\mathbf{a}} \in \text{Cl}(X)} H^0(X, \mathcal{O}(\bar{\mathbf{a}}) \otimes \mathcal{D} \otimes \mathcal{O}(\bar{\mathbf{b}}))$$

is an isomorphism.

The first step is to prove that $\eta_{(*, \bar{\mathbf{b}})}$ induces an isomorphism of the associated sheaves. Fix $\sigma \in \Delta$ and choose $\mathbf{b} \in \mathbb{Z}^d$, mapping to $\bar{\mathbf{b}}$ in $\text{Cl}(X)$, such that $x^{\mathbf{b}}$ is an invertible element in $S_{x\hat{\sigma}}$. Since the restriction of $\mathcal{O}(\bar{\mathbf{b}})$ to U_σ is trivial, one may always find such a \mathbf{b} . We then have

a commutative diagram:

$$\begin{array}{ccc} \frac{(A_{x\hat{\sigma}})_{\bar{\mathbf{b}}}}{(A_{x\hat{\sigma}})_{\bar{\mathbf{b}}}, (\theta_{\bar{\mathbf{u}}}: \bar{\mathbf{u}} \in \text{Cl}(X)^\vee)} & \xrightarrow{\bar{\varphi}_\sigma} & H^0(U_\sigma, \mathcal{D}) \\ \cdot x^{\mathbf{b}} \downarrow & & \downarrow \otimes x^{\mathbf{b}} \\ \frac{(A_{x\hat{\sigma}})_{\bar{\mathbf{b}}}}{(A_{x\hat{\sigma}})_{\bar{\mathbf{b}}}, (\theta_{\bar{\mathbf{u}} + \langle \bar{\mathbf{u}}, \mathbf{b} \rangle}: \bar{\mathbf{u}} \in \text{Cl}(X)^\vee)} & \xrightarrow{\bar{\varphi}'_\sigma} & H^0(U_\sigma, \mathcal{D} \otimes \mathcal{O}(\bar{\mathbf{b}})) \end{array}$$

where $\bar{\varphi}_\sigma$ is the morphism in Proposition 3.7 and $\bar{\varphi}'_\sigma$ is the analogous morphism induced by $\eta_{(\bar{\mathbf{0}}, \bar{\mathbf{b}})}$. Now, Proposition 3.7 implies that $\bar{\varphi}_\sigma$ is an isomorphism and the vertical arrows are clearly isomorphisms. It follows that $\bar{\varphi}'_\sigma$ is an isomorphism and therefore $\eta_{(*, \bar{\mathbf{b}})}$ induces an isomorphism of the associated sheaves.

If F is a graded S -module, we write

$$\Gamma_L(\tilde{F}) = \bigoplus_{\bar{\mathbf{a}} \in \text{Cl}(X)} H^0(X, \mathcal{O}(\bar{\mathbf{a}}) \otimes \tilde{F}).$$

For every such F , there is an exact sequence

$$(3.F) \quad 0 \longrightarrow H_{\bar{\mathbf{b}}}^0(F) \longrightarrow F \longrightarrow \Gamma_L(\tilde{F}) \longrightarrow H_{\bar{\mathbf{b}}}^1(F) \longrightarrow 0,$$

where \mathbf{b} is the irrelevant ideal; see [EMS]. Hence, if $H_{\bar{\mathbf{b}}}^0(D_L(\bar{\mathbf{b}}))$ and $H_{\bar{\mathbf{b}}}^1(D_L(\bar{\mathbf{b}}))$ both vanish, then $\eta_{(*, \bar{\mathbf{b}})}$ is an isomorphism. We relegated these vanishing results to Propositions 3.8 and 3.9 below. \square

Our first vanishing result is

Proposition 3.8. *The element $x^{\mathbf{e}} \in \mathbf{b}$ is not a zero divisor on $D_L(\bar{\mathbf{b}})$ and $H_{\bar{\mathbf{b}}}^0(D_L(\bar{\mathbf{b}})) = 0$.*

Proof. The first assertion implies the second, so it suffices to show that $x^{\mathbf{e}}$ is not a zero divisor. Every $\mathbf{a} \in \mathbb{Z}^d$ can be written uniquely as $\mathbf{a} = \mathbf{a}^+ - \mathbf{a}^-$ where \mathbf{a}^+ and \mathbf{a}^- are non-negative and have disjoint support. Consider the action of the torus $(k^*)^d$ on the Weyl algebra A ; the corresponding eigenspace decomposition is $A = \bigoplus_{\mathbf{a} \in \mathbb{Z}^d} x^{\mathbf{a}^+} \partial^{\mathbf{a}^-} \cdot W$, where $W = k[\theta_1, \dots, \theta_d]$. Let $L_0 \subset W$ be the ideal generated by $\theta_{\bar{\mathbf{u}}} + \langle \bar{\mathbf{u}}, \bar{\mathbf{b}} \rangle$ for all $\bar{\mathbf{u}} \in \text{Cl}(X)^\vee$. Since L_0 is generated by linear forms, it is a prime ideal. Let L denote the left A -ideal $(\theta_{\bar{\mathbf{u}}} + \langle \bar{\mathbf{u}}, \bar{\mathbf{b}} \rangle : \bar{\mathbf{u}} \in \text{Cl}(X)^\vee)$. With this notation, we have $L = \bigoplus_{\mathbf{a} \in \mathbb{Z}^d} x^{\mathbf{a}^+} \partial^{\mathbf{a}^-} \cdot L_0$.

For $x^{\mathbf{e}}$ to be a non-zero divisor on $D_L(\bar{\mathbf{b}})$, it suffices to prove that, for $g \in W$ and $\mathbf{a} \in \mathbb{Z}^d$, the relation $x^{\mathbf{e}} x^{\mathbf{a}^+} \partial^{\mathbf{a}^-} g \in L$ implies $g \in L_0$. To

accomplish this, we first note

$$\begin{aligned}
x^{\mathbf{e}} \cdot x^{\mathbf{a}^+} \partial^{\mathbf{a}^-} &= \left(\prod_{\mathbf{a}_i \geq 0} x_i^{\mathbf{a}_i+1} \right) \left(\prod_{\mathbf{a}_i < 0} x_i \partial^{-\mathbf{a}_i} \right) \\
&= \left(\prod_{\mathbf{a}_i \geq 0} x_i^{\mathbf{a}_i+1} \right) \left(\prod_{\mathbf{a}_i < 0} \partial^{-\mathbf{a}_i-1} (\theta_i + \mathbf{a}_i + 1) \right) \\
&= x^{(\mathbf{a}+\mathbf{e})^+} \partial^{(\mathbf{a}+\mathbf{e})^-} \prod_{\mathbf{a}_i < 0} (\theta_i + \mathbf{a}_i + 1).
\end{aligned}$$

Hence, $x^{\mathbf{e}} \cdot x^{\mathbf{a}^+} \partial^{\mathbf{a}^-} g \in L$ implies $(\prod_{\mathbf{a}_i < 0} (\theta_i + \mathbf{a}_i + 1)) \cdot g \in L_0$. Suppose $g \notin L_0$. Since L_0 is a prime ideal, there exists an index i such that $\mathbf{a}_i < 0$ and $\theta_i + \mathbf{a}_i + 1 \in L_0$. Expressing $\theta_i + \mathbf{a}_i + 1$ in terms of the linear generators of L_0 , it follows that there is $\bar{\mathbf{u}} \in \text{Cl}(X)^\vee$ such that $\theta_i = \theta_{\bar{\mathbf{u}}}$. However, for every $\mathbf{w} \in N^\vee$, equation (2.A) implies

$$\langle \bar{\mathbf{u}}, \langle \mathbf{w}, \mathbf{v}_1 \rangle \bar{\mathbf{e}}_1 + \cdots + \langle \mathbf{w}, \mathbf{v}_d \rangle \bar{\mathbf{e}}_d \rangle = 0,$$

from which we deduce that $\langle \mathbf{w}, \mathbf{v}_i \rangle = 0$ and $\mathbf{v}_i = 0$ giving a contradiction. \square

We end this section with

Proposition 3.9. *The local cohomology module $H_{\mathfrak{b}}^1(D_L(\bar{\mathfrak{b}}))$ vanishes.*

Proof. Since $x^{\mathbf{e}}$ is not a zero divisor on $D_L(\bar{\mathfrak{b}})$, there is a short exact sequence of S -modules

$$0 \longrightarrow D_L(\bar{\mathfrak{b}}) \xrightarrow{x^{\mathbf{e}}} D_L(\bar{\mathfrak{b}}) \longrightarrow Q = \frac{D_L(\bar{\mathfrak{b}})}{x^{\mathbf{e}} \cdot D_L(\bar{\mathfrak{b}})} \longrightarrow 0,$$

and the long exact sequence of local cohomology gives

$$(3.G) \quad 0 \longrightarrow H_{\mathfrak{b}}^0(Q) \longrightarrow H_{\mathfrak{b}}^1(D_L(\bar{\mathfrak{b}})) \xrightarrow{x^{\mathbf{e}}} H_{\mathfrak{b}}^1(D_L(\bar{\mathfrak{b}})).$$

Because every element in $H_{\mathfrak{b}}^1(D_L(\bar{\mathfrak{b}}))$ is annihilated by a power of \mathfrak{b} , the injectivity of $x^{\mathbf{e}} \cdot$ in equation (3.G) implies that $H_{\mathfrak{b}}^1(D_L(\bar{\mathfrak{b}})) = 0$. Thus, it suffices to prove that $H_{\mathfrak{b}}^0(Q) = 0$.

Let K be the left A -ideal satisfying $Q = A(\bar{\mathfrak{b}})/K$. To prove that $H_{\mathfrak{b}}^0(Q) = 0$, we must show that if $f \in A$ satisfies $(x^{\bar{\sigma}})^m f \in K$ for some $m \geq 1$ and all $\sigma \in \Delta$, then $f \in K$. Using the notation from the proof of Proposition 3.8, we have $K = L + x^{\mathbf{e}} \cdot A$. From the decomposition

of A , we have

$$\begin{aligned} x^{\mathbf{e}} \cdot A &= \bigoplus_{\mathbf{a} \in \mathbb{Z}^d} x^{\mathbf{e}} \cdot x^{\mathbf{a}^+} \partial^{\mathbf{a}^-} \cdot W \\ &= \bigoplus_{\mathbf{a} \in \mathbb{Z}^d} \left(x^{(\mathbf{a}+\mathbf{e})^+} \partial^{(\mathbf{a}+\mathbf{e})^-} \left(\prod_{\mathbf{a}_i < 0} (\theta_i + \mathbf{a}_i + 1) \right) \cdot W \right), \end{aligned}$$

and we deduce $K = \bigoplus_{\mathbf{a} \in \mathbb{Z}^d} x^{\mathbf{a}^+} \partial^{\mathbf{a}^-} \cdot K(\mathbf{a})$, where $K(\mathbf{a})$ is defined to be $L_0 + \left(\prod_{\mathbf{a}_i \leq 0} (\theta_i + \mathbf{a}_i) \right) \cdot W$. Thus, it is enough to consider elements $f \in A$ of the form $x^{\mathbf{a}^+} \partial^{\mathbf{a}^-} g$ with $g \in W$ and prove that $g \in K(\mathbf{a})$. Moreover, we may assume that $m + \mathbf{a}_i > 0$ for $1 \leq i \leq d$.

By induction on r , we see that $x_i^r \partial_i^r = \prod_{j=1}^r (\theta_i - j + 1)$ for $r \geq 1$. Hence, for $\mathbf{a}_i < 0$, we have

$$x_i^m \partial_i^{-\mathbf{a}_i} = x_i^{m+\mathbf{a}_i} x_i^{-\mathbf{a}_i} \partial_i^{-\mathbf{a}_i} = x_i^{m+\mathbf{a}_i} \cdot \prod_{j=1}^{-\mathbf{a}_i} (\theta_i - j + 1)$$

and, for $\sigma \in \Delta$, we obtain

$$(x^{\hat{\sigma}})^m x^{\mathbf{a}^+} \partial^{\mathbf{a}^-} = \left(x^{(\mathbf{a}+m\hat{\sigma})^+} \partial^{(\mathbf{a}+m\hat{\sigma})^-} \right) \cdot \prod_{i \in \hat{\Lambda}} \prod_{j=1}^{-\mathbf{a}_i} (\theta_i - j + 1),$$

where $\hat{\Lambda} = \{i : \mathbf{v}_i \notin \sigma \text{ and } \mathbf{a}_i < 0\}$. We deduce

$$(3.H) \quad \left(\prod_{i \in \hat{\Lambda}} \prod_{j=1}^{-\mathbf{a}_i} (\theta_i - j + 1) \right) \cdot g \in L_0 + \left(\prod_{i \in \Lambda} (\theta_i + \mathbf{a}_i) \right) \cdot W,$$

where $\Lambda = \{i : \mathbf{v}_i \in \sigma \text{ and } \mathbf{a}_i \leq 0\}$.

For each $\mathbf{b} \in \mathbb{Z}^d$, we define an automorphism $\alpha_{\mathbf{b}} : W \rightarrow W$ given by $\alpha_{\mathbf{b}}(\theta_i) = \theta_i - \mathbf{b}_i$ and we define $\rho_{\mathbf{b}} : W \rightarrow W'$ to be the composition $\rho_{\mathbf{b}} = \rho \circ \alpha_{\mathbf{b}}$ (the map ρ is defined in the paragraph before Lemma 3.5). It is clear that $\rho_{\mathbf{b}}$ is surjective and $\text{Ker}(\rho_{\mathbf{b}}) = L_0$. Applying $\rho_{\mathbf{b}}$ to equation (3.H) gives

$$\left(\prod_{i \in \hat{\Lambda}} \prod_{j=1}^{-\mathbf{a}_i} \rho_{\mathbf{b}}(\theta_i - j + 1) \right) \cdot \rho_{\mathbf{b}}(g) \in \left(\prod_{i \in \Lambda} \rho_{\mathbf{b}}(\theta_i + \mathbf{a}_i) \right) \cdot W'.$$

Lemma 3.6 obviously extends to $\rho_{\mathbf{b}}$ and implies

$$\rho_{\mathbf{b}}(g) \in \left(\prod_{i \in \Lambda} \rho_{\mathbf{b}}(\theta_i + \mathbf{a}_i) \right) \cdot W'.$$

Since this relation holds for every $\sigma \in \Delta$, a second application of Lemma 3.6 shows that $\rho_{\mathbf{b}}(g) \in \left(\prod_{\mathbf{a}_i \leq 0} \rho_{\mathbf{b}}(\theta_i + \mathbf{a}_i)\right) \cdot W'$ and therefore $g \in K(\mathbf{a}) = L_0 + \left(\prod_{\mathbf{a}_i \leq 0} (\theta_i + \mathbf{a}_i)\right) \cdot W$. \square

4. \mathcal{D} -MODULES

We now use Theorem 3.7 to describe the relation between A -modules and \mathcal{D} -modules on X . We begin by showing that the S -module associated to a \mathcal{D} -module has a graded A -module structure.

Proposition 4.1. *If \mathcal{F} is a left \mathcal{D} -module, then the graded S -module*

$$\Gamma_L(\mathcal{F}) = \bigoplus_{\bar{\mathbf{a}} \in \text{Cl}(X)} H^0(X, \mathcal{O}(\bar{\mathbf{a}}) \otimes \mathcal{F}),$$

has a graded left A -module structure, extending the left S -module structure. Similarly, if \mathcal{G} is a right \mathcal{D} -module, then the graded S -module

$$\Gamma_R(\mathcal{G}) = \bigoplus_{\bar{\mathbf{b}} \in \text{Cl}(X)} H^0(X, \mathcal{G} \otimes \mathcal{O}(\bar{\mathbf{b}})),$$

has a graded right A -module structure, extending the right S -module structure.

Proof. We present the left \mathcal{D} -modules case here — the proof for right \mathcal{D} -modules is completely analogous. For the first assertion, it is enough as well to construct k -linear maps

$$\mu_{\bar{\mathbf{a}}', \bar{\mathbf{a}}}^{\mathcal{F}}: A_{\bar{\mathbf{a}}'} \otimes_k H^0(X, \mathcal{O}(\bar{\mathbf{a}}) \otimes \mathcal{F}) \longrightarrow H^0(X, \mathcal{O}(\bar{\mathbf{a}} + \bar{\mathbf{a}}') \otimes \mathcal{F}),$$

for all $\bar{\mathbf{a}}, \bar{\mathbf{a}}' \in \text{Cl}(X)$, satisfying the obvious axioms. To accomplish this, we consider a local version of the left multiplication map (3.A) when $\bar{\mathbf{b}} = 0$. More explicitly, for each $\sigma \in \Delta$, this morphism is

$$\mu_{\bar{\mathbf{a}}', \bar{\mathbf{a}}}^{\mathcal{F}}|_{U_\sigma}: A_{\bar{\mathbf{a}}'} \otimes_k H^0(U_\sigma, \mathcal{O}(\bar{\mathbf{a}}) \otimes \mathcal{D}) \longrightarrow H^0(U_\sigma, \mathcal{O}(\bar{\mathbf{a}} + \bar{\mathbf{a}}') \otimes \mathcal{D}),$$

given by $\mu_{\bar{\mathbf{a}}', \bar{\mathbf{a}}}^{\mathcal{F}}|_{U_\sigma}(f \otimes s) = f \circ s$. We claim that $\mu_{\bar{\mathbf{a}}', \bar{\mathbf{a}}}^{\mathcal{F}}|_{U_\sigma}$ is a morphism of right $H^0(U_\sigma, \mathcal{D})$ -modules. To see this, recall that sections $s \in H^0(U_\sigma, \mathcal{O}(\bar{\mathbf{b}}) \otimes \mathcal{D})$ and $g \in H^0(U_\sigma, \mathcal{D})$ can be identified with elements in $\text{Hom}_k((S_{x\hat{\sigma}})_0, (S_{x\hat{\sigma}})_{\bar{\mathbf{b}}})$ and $\text{Hom}_k((S_{x\hat{\sigma}})_0, (S_{x\hat{\sigma}})_0)$ respectively. In particular, we have

$$(\mu_{\bar{\mathbf{a}}', \bar{\mathbf{a}}}^{\mathcal{F}}|_{U_\sigma}(f \otimes s)) \cdot g = f \circ s \circ g = \mu_{\bar{\mathbf{a}}', \bar{\mathbf{a}}}^{\mathcal{F}}|_{U_\sigma}(f \otimes s \cdot g),$$

for all $f \in A_{\bar{\mathbf{a}}'}$. It follows that, by tensoring the map $\mu_{\bar{\mathbf{a}}', \bar{\mathbf{a}}}^{\mathcal{F}}|_{U_\sigma}$ on the right with $H^0(U_\sigma, \mathcal{F})$ over $H^0(U_\sigma, \mathcal{D})$, we obtain a k -linear map

$$\mu_{\bar{\mathbf{a}}', \bar{\mathbf{a}}}^{\mathcal{F}}|_{U_\sigma}: A_{\bar{\mathbf{a}}'} \otimes_k H^0(U_\sigma, \mathcal{O}(\bar{\mathbf{a}}) \otimes \mathcal{F}) \longrightarrow H^0(U_\sigma, \mathcal{O}(\bar{\mathbf{a}} + \bar{\mathbf{a}}') \otimes \mathcal{F}).$$

These maps glue together to give $\mu_{\bar{\mathbf{a}}', \bar{\mathbf{a}}}^{\mathcal{F}}$ which makes F into a graded left A -module. \square

Let $A\text{-GrMod}_\theta$ be the category of graded left A -modules F such that

$$(4.A) \quad (\theta_{\bar{\mathbf{u}}} - \langle \bar{\mathbf{u}}, \bar{\mathbf{a}} \rangle) \cdot F_{\bar{\mathbf{a}}} = 0 \text{ for all } \bar{\mathbf{a}} \in \text{Cl}(X) \text{ and all } \bar{\mathbf{u}} \in \text{Cl}(X)^\vee.$$

A graded A -module F is called \mathfrak{b} -torsion if, for every $f \in F$, there exists $\ell > 0$ such that $\mathfrak{b}^\ell f = 0$. Let $\mathfrak{b}\text{-Tors}$ denote the full subcategory of \mathfrak{b} -torsion modules. Similarly, $\text{GrMod}_\theta\text{-}A$ is the category of graded right A -modules G satisfying $G_{\bar{\mathbf{b}}} \cdot (\theta_{\bar{\mathbf{u}}} + \langle \bar{\mathbf{u}}, \bar{\mathbf{b}} \rangle) = 0$ for all $\bar{\mathbf{b}} \in \text{Cl}(X)$ and all $\bar{\mathbf{u}} \in \text{Cl}(X)^\vee$. Let $\text{Tors-}\mathfrak{b}$ denote the full subcategory of \mathfrak{b} -torsion modules in $\text{GrMod}_\theta\text{-}A$. It is clear that $A\text{-GrMod}_\theta$ and $\text{GrMod}_\theta\text{-}A$ are both abelian categories closed under taking graded subquotients. The main result in this section is:

Theorem 4.2. *The map $F \mapsto \tilde{F}$ is an exact functor from $A\text{-GrMod}_\theta$ to $\mathcal{D}\text{-Mod}$, $\mathcal{F} \mapsto \Gamma_L(\mathcal{F})$ is a left exact functor from $\mathcal{D}\text{-Mod}$ to $A\text{-GrMod}_\theta$ and there are natural transformations of functors*

$$\text{id}_{A\text{-GrMod}_\theta/\mathfrak{b}\text{-Tors}} \xrightarrow{\cong} \Gamma_L \circ \sim \quad \text{and} \quad \sim \circ \Gamma_L \xrightarrow{\cong} \text{id}_{\mathcal{D}\text{-Mod}}.$$

Similarly, the map $G \mapsto \tilde{G}$ is an exact functor from $\text{GrMod}_\theta\text{-}A$ to $\text{Mod-}\mathcal{D}$, $\mathcal{G} \mapsto \Gamma_R(\mathcal{G})$ is a left exact functor from $\text{Mod-}\mathcal{D}$ to $\text{GrMod}_\theta\text{-}A$ and there are natural transformations of functors

$$\text{id}_{\text{GrMod}_\theta\text{-}A/\text{Tors-}\mathfrak{b}} \xrightarrow{\cong} \Gamma_R \circ \sim \quad \text{and} \quad \sim \circ \Gamma_R \xrightarrow{\cong} \text{id}_{\text{Mod-}\mathcal{D}}.$$

In particular, every left \mathcal{D} -module is of the form \tilde{F} for some graded left A -module F and every right \mathcal{D} -module is of the form \tilde{G} for some graded right A -module G .

Proof of Theorem 4.2. Again, we give the proof only for left modules. For the first part, we consider an object F in $A\text{-GrMod}_\theta$. By definition, we have $H^0(U_\sigma, \tilde{F}) = (F_{x^\sigma})_{\bar{\mathbf{0}}}$, where $(F_{x^\sigma})_{\bar{\mathbf{0}}}$ is a left $(A_{x^\sigma})_{\bar{\mathbf{0}}}$ -module and $\sigma \in \Delta$. In light of Theorem 3.7, we must show that, for every $\bar{\mathbf{u}} \in \text{Cl}(X)^\vee$, we have $\theta_{\bar{\mathbf{u}}} \cdot (F_{x^\sigma})_{\bar{\mathbf{0}}} = 0$. Now, if $\frac{f}{(x^\sigma)^m} \in (F_{x^\sigma})_{\bar{\mathbf{0}}}$, then $f \in F_{m\bar{\mathbf{e}}_\sigma}$ where $\mathbf{e}_\sigma = \sum_{i \in \hat{\sigma}} \mathbf{e}_i$ and the hypotheses on F imply

$$\begin{aligned} \theta_{\bar{\mathbf{u}}} \cdot \frac{f}{(x^\sigma)^m} &= \left(\theta_{\bar{\mathbf{u}}} \frac{1}{(x^\sigma)^m} \right) \cdot f \\ &= \frac{1}{(x^\sigma)^m} \theta_{\bar{\mathbf{u}}} \cdot f - \sum_{i \in \hat{\sigma}} m \langle \bar{\mathbf{u}}, \bar{\mathbf{e}}_i \rangle \frac{1}{(x^\sigma)^m} \cdot f \\ &= \frac{1}{(x^\sigma)^m} (\theta_{\bar{\mathbf{u}}} - \langle \bar{\mathbf{u}}, m\bar{\mathbf{e}}_\sigma \rangle) \cdot f = 0. \end{aligned}$$

Therefore $H^0(U_\sigma, \tilde{F})$ has a structure of left module over $H^0(U_\sigma, \mathcal{D})$ for every $\sigma \in \Delta$. It is straightforward to verify that these structures glue together to give a \mathcal{D} -module structure on \tilde{F} . By construction, we see that \tilde{F} is quasi-coherent sheaf over \mathcal{D} .

Conversely, let \mathcal{F} be an object of $\mathcal{D}\text{-Mod}$. Applying Proposition 4.1, we know that $F = \Gamma_L(\mathcal{F})$ is a graded left A -module, so it is enough to prove that F satisfies (4.A). Fixing $\bar{\mathbf{u}} \in \text{Cl}(X)^\vee$ and $\bar{\mathbf{a}} \in \text{Cl}(X)$, it suffices to show that

$$\mu_{\bar{\mathbf{0}}, \bar{\mathbf{a}}}^{\mathcal{F}}|_{U_\sigma}((\theta_{\bar{\mathbf{u}}} - \langle \bar{\mathbf{u}}, \bar{\mathbf{a}} \rangle) \otimes s') = 0,$$

for every $\sigma \in \Delta$ and all sections $s' \in H^0(U_\sigma, \mathcal{O}(\bar{\mathbf{a}}) \otimes \mathcal{F})$. As explained in Proposition 4.1, we have

$$\begin{aligned} H^0(U_\sigma, \mathcal{O}(\bar{\mathbf{a}}) \otimes \mathcal{F}) &= H^0(U_\sigma, \mathcal{O}(\bar{\mathbf{a}}) \otimes \mathcal{D} \otimes_{\mathcal{D}} \mathcal{F}) \\ &= H^0(U_\sigma, \mathcal{O}(\bar{\mathbf{a}}) \otimes \mathcal{D}) \otimes_{H^0(U_\sigma, \mathcal{D})} H^0(U_\sigma, \mathcal{F}), \end{aligned}$$

so that we may identify s' with a linear combination of elements of the form $s \otimes f$. By definition, we have

$$\mu_{\bar{\mathbf{0}}, \bar{\mathbf{a}}}^{\mathcal{F}}|_{U_\sigma}((\theta_{\bar{\mathbf{u}}} - \langle \bar{\mathbf{u}}, \bar{\mathbf{a}} \rangle) \otimes s \otimes f) = (\theta_{\bar{\mathbf{u}}} - \langle \bar{\mathbf{u}}, \bar{\mathbf{a}} \rangle) \circ s \otimes f,$$

and we claim that $(\theta_{\bar{\mathbf{u}}} - \langle \bar{\mathbf{u}}, \bar{\mathbf{a}} \rangle) \circ s$ is zero. Indeed, for every $x^c \in (S_{x^{\hat{\sigma}}})_{\bar{\mathbf{a}}}$, we have $\theta_{\bar{\mathbf{u}}}(x^c) = \langle \bar{\mathbf{u}}, \bar{\mathbf{e}}_i \rangle c_1 x^c + \cdots + \langle \bar{\mathbf{u}}, \bar{\mathbf{e}}_i \rangle c_d x^c = \langle \bar{\mathbf{u}}, \bar{\mathbf{a}} \rangle x^c$. Therefore, if $g \in H^0(U_\sigma, \mathcal{O}) = (S_{x^{\hat{\sigma}}})_0$, then $s(g) \in (S_{x^{\hat{\sigma}}})_{\bar{\mathbf{a}}}$ and

$$(\theta_{\bar{\mathbf{u}}} - \langle \bar{\mathbf{u}}, \bar{\mathbf{a}} \rangle)(s(g)) = \langle \bar{\mathbf{u}}, \bar{\mathbf{a}} \rangle s(g) - \langle \bar{\mathbf{u}}, \bar{\mathbf{a}} \rangle s(g) = 0.$$

Finally, the exact sequence (3.F) provides the first natural transformation, once we observe that $H_{\mathfrak{b}}^i(F)$ is \mathfrak{b} -torsion. It follows from Cox [C] that the sheaf associated to $\Gamma_L(\mathcal{F})$ is isomorphic to \mathcal{F} . \square

As a corollary, we obtain

Proof of Theorem 1.1. Follows immediately from Theorem 4.2. \square

We next turn our attention to coherent \mathcal{D} -modules and finitely generated graded A -modules. We write $A\text{-GrMod}_\theta^f$ and $\text{GrMod}_\theta^f\text{-}A$ for the full subcategories of $A\text{-GrMod}_\theta$ and $\text{GrMod}_\theta\text{-}A$ consisting of finitely generated A -modules.

Proposition 4.3. *If F is an object in $A\text{-GrMod}_\theta^f$ then \tilde{F} is a coherent left \mathcal{D} -module. Moreover, every coherent left \mathcal{D} -module is of the form \tilde{F} for some $F \in A\text{-GrMod}_\theta^f$. Similarly, $G \in \text{GrMod}_\theta^f\text{-}A$ implies \tilde{G} belongs to $\text{Coh-}\mathcal{D}$ and every coherent right \mathcal{D} -module is isomorphic to \tilde{G} for some $G \in \text{GrMod}_\theta^f\text{-}A$.*

Our proof is analogous to the \mathcal{O} -module case found in Cox [C].

Proof. Once again, we present the proof only for left modules. Suppose that F belongs to $A\text{-GrMod}_\theta^f$. To establish that $\tilde{F} \in \mathcal{D}\text{-Coh}$, we have to check that, for every $\sigma \in \Delta$, $(F_{x^{\hat{\sigma}}})_{\bar{\mathbf{0}}}$ is finitely generated over $(A_{x^{\hat{\sigma}}})_{\bar{\mathbf{0}}}$. Thus, it suffices to show that, for every element $f \in F$, there is an

invertible element $g \in S_{x^{\hat{\sigma}}}$ such that $g \cdot f$ has degree zero. Consider f in $F_{\bar{\mathbf{a}}}$. Since X is smooth, there is a divisor corresponding to $\mathbf{a} \in \mathbb{Z}^d$ supported outside σ and whose class is $\bar{\mathbf{a}}$. It follows that $x^{-\mathbf{a}}$ is an invertible element in $S_{x^{\hat{\sigma}}}$ and $x^{-\mathbf{a}} \cdot f$ has degree zero.

For the second assertion, we must show that given a coherent left \mathcal{D} -module \mathcal{F} there exists a finitely generated A -submodule F of $\Gamma_L(\mathcal{F})$ such that $\tilde{F} \cong \mathcal{F}$. Since \mathcal{F} is coherent, $H^0(U_\sigma, \mathcal{F})$ is finitely generated over $H^0(U_\sigma, \mathcal{D})$, for every $\sigma \in \Delta$. Choose, for each $\sigma \in \Delta$, a finite set of homogeneous elements in $\Gamma_L(\mathcal{F})$ which are the numerators for a corresponding set of generators of $H^0(U_\sigma, \mathcal{F})$. Setting F to be the A -submodule of $\Gamma_L(\mathcal{F})$ generated by the union of these sets, we have $\tilde{F} \cong \mathcal{F}$ and F is finitely generated over A . \square

Remark 4.4. As a consequence of Theorem 4.2, we have $\tilde{F} = 0$ if and only if the graded S -module F satisfies $F = H_{\mathfrak{b}}^0(F)$; this is equivalent to saying that F is a \mathfrak{b} -torsion module. At the other extreme, F has no \mathfrak{b} -torsion when $H_{\mathfrak{b}}^0(F) = 0$ and we say that F is \mathfrak{b} -saturated when $H_{\mathfrak{b}}^0(F) = H_{\mathfrak{b}}^1(F) = 0$. Now, every left \mathcal{D} -module \mathcal{F} can be represented by a unique saturated A -module, namely $\Gamma_L(\mathcal{F})$. Unfortunately, this may not be finitely generated, even if \mathcal{F} is coherent. However, by replacing F with a suitable submodule of $F/H_{\mathfrak{b}}^0(F)$, we may assume that F has no \mathfrak{b} -torsion and is finitely generated, whenever \mathcal{F} is coherent.

Example 4.5. The A -module corresponding to the structure sheaf \mathcal{O} (which is a left \mathcal{D} -module) is $\Gamma_L(\mathcal{O}) = \frac{A}{A \cdot (\partial_1, \dots, \partial_d)}$. In particular, $\Gamma_L(\mathcal{O})$ is isomorphic to S , where S has the standard A -module structure.

Corollary 4.6. *For every $\bar{\mathbf{b}} \in \text{Cl}(X)$, there is an isomorphism of graded left A -modules $\Gamma_L(\mathcal{D} \otimes \mathcal{O}(\bar{\mathbf{b}})) \cong D_L(\bar{\mathbf{b}})$. Similarly, we have $\Gamma_R(\mathcal{O}(\bar{\mathbf{a}}) \otimes \mathcal{D}) \cong D_R(\bar{\mathbf{a}})$.*

Proof. Theorem 3.4 provides $\eta_{(*, \bar{\mathbf{0}})}: D_L(\bar{\mathbf{b}}) \xrightarrow{\cong} \Gamma_L(\mathcal{D} \otimes \mathcal{O}(\bar{\mathbf{b}}))$ and $\eta_{(\bar{\mathbf{0}}, *)}: D_R(\bar{\mathbf{a}}) \xrightarrow{\cong} \Gamma_R(\mathcal{O}(\bar{\mathbf{a}}) \otimes \mathcal{D})$. \square

Corollary 4.7. *For $\bar{\mathbf{a}}, \bar{\mathbf{b}} \in \text{Cl}(X)$, we have $D_L(\bar{\mathbf{a}}) \in A\text{-GrMod}_\theta$ and $D_R(\bar{\mathbf{b}}) \in \text{GrMod}_{\theta-A}$.*

Proof. This follows from Corollary 4.6 and Theorem 4.2. \square

Corollary 4.8. *Let F be a graded left A -module generated by homogeneous elements $\{f_i\}_i$ with $\deg f_i = \bar{\mathbf{b}}_i$. For F to belong to $A\text{-GrMod}_\theta$, it is necessary and sufficient that $(\theta_{\bar{\mathbf{u}}} - \langle \bar{\mathbf{u}}, \bar{\mathbf{b}}_i \rangle) \cdot f_i = 0$ for all i and all $\bar{\mathbf{u}} \in \text{Cl}(X)^\vee$. A similar assertion holds for graded right A -modules.*

Proof. This condition is clearly necessary. To see the other direction, consider the surjective graded morphism defined by the given generators: $\bigoplus_i A(-\bar{\mathbf{b}}_i) \longrightarrow F$. By hypothesis, this factors to an epimorphism $\bigoplus_i D_L(-\bar{\mathbf{b}}_i) \longrightarrow F$ and Corollary 4.7 implies that $F \in A\text{-GrMod}_\theta$. \square

Corollary 4.9. *For each $\mathcal{F} \in \mathcal{D}\text{-Mod}$, there exist a family $\{\bar{\mathbf{b}}_i\}_i$ of elements in $\text{Cl}(X)$ and an epimorphism $\bigoplus_i \mathcal{D} \otimes \mathcal{O}(-\bar{\mathbf{b}}_i) \longrightarrow \mathcal{F}$. The analogous result also holds for $\mathcal{G} \in \text{Mod-}\mathcal{D}$.*

Proof. Theorem 4.2 implies that there exists $F \in A\text{-GrMod}_\theta$ such that $\mathcal{F} \cong \tilde{F}$. Now, Corollary 4.8 gives an epimorphism $\bigoplus_i D_L(-\bar{\mathbf{b}}_i) \longrightarrow F$. Taking the corresponding morphism of sheaves and applying Corollary 4.6 establishes the claim. \square

Remark 4.10. Applying Corollary 4.9, one can construct a resolution of a \mathcal{D} -module using twisted modules $\mathcal{D} \otimes \mathcal{O}(\bar{\mathbf{b}}_i)$. More concretely, given a graded module F satisfying $\tilde{F} \cong \mathcal{F}$, Gröbner basis techniques can be used to construct a resolution of F by modules $D_L(\bar{\mathbf{b}})$ which will then lift to a resolution of \mathcal{F} . It would be interesting to investigate the relationship between these two types of resolutions.

The last part of this section is devoted to the categorical equivalence between right and left \mathcal{D} -modules on the toric variety X . Recall that, for a smooth variety X of dimension n , the sheaf of differential forms of top degree Ω^n has a natural structure of right \mathcal{D} -module extending the usual \mathcal{O} -module structure. Locally, right multiplication of an n -form ω with a vector field ν is defined by $\omega \cdot \nu = -\text{Lie}_\nu(\omega)$, where $\text{Lie}_\nu(\omega)$ is the Lie derivative of ω along ν (see [B⁺] Chapter VI.3.4 for details).

The equivalence of categories $\tau_{LR}: \mathcal{D}\text{-Mod} \longrightarrow \text{Mod-}\mathcal{D}$ with inverse $\tau_{RL}: \text{Mod-}\mathcal{D} \longrightarrow \mathcal{D}\text{-Mod}$ is defined as follows: For a left \mathcal{D} -module \mathcal{F} , we have $\tau_{LR}(\mathcal{F}) = \mathcal{F} \otimes \Omega^n$ where the right multiplication with a vector field ν is $(f \otimes \omega) \cdot \nu = -\nu(f) \otimes \omega + f \otimes \omega \cdot \nu$. Similarly, if \mathcal{G} is a right \mathcal{D} -module, then $\tau_{RL}(\mathcal{G}) = \mathcal{H}om_{\mathcal{O}}(\Omega^n, \mathcal{G})$ and left multiplication with a vector field ν is given by $\nu \cdot \psi(\omega) = \psi(\omega \cdot \nu) - \psi(\omega) \cdot \nu$.

In particular, the left–right equivalence of A -modules is given by the algebra involution $\tau: A \longrightarrow A$, where $x^{\mathbf{a}} \partial^{\mathbf{b}} \mapsto (-\partial)^{\mathbf{b}} x^{\mathbf{a}}$. Specifically, given a graded left A -module F , we obtain a graded right A -module F^τ which has the same underlying additive structure and has multiplication defined by $g \cdot f = \tau(f) \cdot g$ for $g \in A$ and $f \in F$. Similarly, if G is a graded right A -module, then an analogous procedure yields the left A -module G^τ . It is clear that $(F^\tau)^\tau = F$ and $(G^\tau)^\tau = G$. Furthermore, for graded A -modules, we have

Proposition 4.11. *There are inverse equivalences of categories*

$$\tau_{LR}^{\text{mod}}: A\text{-GrMod}_\theta \longrightarrow \text{GrMod}_{\theta-A}, \quad \tau_{RL}^{\text{mod}}: \text{GrMod}_{\theta-A} \longrightarrow A\text{-GrMod}_\theta$$

given by $\tau_{LR}^{\text{mod}}(F) = F^\tau(-\bar{\mathbf{e}})$ and $\tau_{RL}^{\text{mod}}(G) = G^\tau(\bar{\mathbf{e}})$ where $\bar{\mathbf{e}} \in \text{Cl}(X)$ is the class of $\mathbf{e} = \mathbf{e}_1 + \cdots + \mathbf{e}_d \in \mathbb{Z}^d$.

Proof. We only need to show that the graded components of $\tau_{LR}^{\text{mod}}(F)$ and $\tau_{RL}^{\text{mod}}(G)$ are annihilated by suitable Euler operators. However, this follows from the fact that $\tau(\theta_{\bar{\mathbf{u}}}) = -\theta_{\bar{\mathbf{u}}} - \langle \bar{\mathbf{u}}, \bar{\mathbf{e}} \rangle$, where $\bar{\mathbf{u}} \in \text{Cl}(X)$. \square

Example 4.12. Since $\tau(\theta_{\bar{\mathbf{u}}} + \langle \bar{\mathbf{u}}, \bar{\mathbf{a}} \rangle) = -(\theta_{\bar{\mathbf{u}}} - \langle \bar{\mathbf{u}}, \bar{\mathbf{a}} - \bar{\mathbf{e}} \rangle)$, there is an isomorphism $\tau_{LR}^{\text{mod}}(D_L(\bar{\mathbf{a}})) \cong D_R(\bar{\mathbf{a}} - \bar{\mathbf{e}})$.

We next show that these equivalences of categories are compatible with the functors in Theorem 4.2. We will use the fact that, for a smooth toric variety X , there is a natural isomorphism $\Omega^n \cong \mathcal{O}(-\bar{\mathbf{e}})$. In fact, if 0 denotes the unique zero dimensional cone in Δ , then this isomorphism identifies the section $\frac{dy_1}{y_1} \wedge \cdots \wedge \frac{dy_n}{y_n}$ with $\frac{1}{x_1 \cdots x_d}$ on the open subset U_0 (see Section 4.3 in [F]).

Proposition 4.13. *For the pair of functors $(\tau_{LR}^{\text{mod}}, \tau_{LR})$, the diagrams*

$$\begin{array}{ccc} A\text{-GrMod}_\theta & \xrightarrow{\tau_{LR}^{\text{mod}}} & \text{GrMod}_{\theta-A} \\ \sim \downarrow & & \sim \downarrow \\ \mathcal{D}\text{-Mod} & \xrightarrow{\tau_{LR}} & \text{Mod-}\mathcal{D} \end{array} \quad \begin{array}{ccc} \mathcal{D}\text{-Mod} & \xrightarrow{\tau_{LR}} & \text{Mod-}\mathcal{D} \\ \Gamma_L \downarrow & & \Gamma_R \downarrow \\ A\text{-GrMod}_\theta & \xrightarrow{\tau_{LR}^{\text{mod}}} & \text{GrMod}_{\theta-A} \end{array}$$

are commutative, up to natural isomorphisms. A similar statement holds for the pair $(\tau_{RL}^{\text{mod}}, \tau_{RL})$.

Proof. Since $\tau_{RL} = (\tau_{LR})^{-1}$ and $\tau_{RL}^{\text{mod}} = (\tau_{LR}^{\text{mod}})^{-1}$, the second assertion is a consequence of the first. Because $F^\tau \cong F$ as S -modules and $\Omega^n \cong \mathcal{O}(-\bar{\mathbf{e}})$ as \mathcal{O} -modules, there is a natural isomorphism of \mathcal{O} -modules $\beta_F: \tau_{LR}(\tilde{F}) = \tilde{F} \otimes \Omega^n \longrightarrow \widetilde{\tau_{LR}^{\text{mod}}(F)} = \widetilde{F^\tau(-\bar{\mathbf{e}})}$. Thus, it suffices to prove that β_F is compatible with the right \mathcal{D} -module structures.

By taking a presentation $\bigoplus_i D_L(\bar{\mathbf{b}}_i) \longrightarrow F$, we see that it suffices to establish the claim for $F = D_L(\bar{\mathbf{b}}_i)$. In this case, the restriction map

$$H^0(U, \widetilde{F^\tau(-\bar{\mathbf{e}})}) \longrightarrow H^0(U', \widetilde{F^\tau(-\bar{\mathbf{e}})}),$$

is injective for open subsets $U' \subseteq U \subseteq X$. Thus, the claim reduces to showing that β_F is compatible with the right \mathcal{D} -module structure on U_0 . Over U_0 , the map

$$\beta_F|_{U_0}: (F_{x_1 \cdots x_d})_{\bar{\mathbf{0}}} \otimes_{(S_{x_1 \cdots x_d})_{\bar{\mathbf{0}}}} H^0(U_0, \Omega^n) \longrightarrow (F_{x_1 \cdots x_d})_{-\bar{\mathbf{e}}}^\tau$$

is given by $f \otimes \omega \mapsto \frac{f}{x_1 \cdots x_d}$, where $\omega = \frac{dy_1}{y_1} \wedge \cdots \wedge \frac{dy_n}{y_n}$. Now, it is enough to check that $\beta_F|_{U_0}$ is compatible with right multiplication with a vector field ν over U_0 . Using the notation from Lemma 3.5, we may assume that $\nu = y^{\mathbf{p}}\rho(\theta_i)$, for some $\mathbf{p} \in \mathbb{Z}^n$. We first compute

$$(f \otimes \omega) \cdot y^{\mathbf{p}}\rho(\theta_i) = -(y^{\mathbf{p}}\rho(\theta_i) \cdot f) \otimes \omega + f \otimes (\omega \cdot y^{\mathbf{p}}\rho(\theta_i)).$$

By definition, we have

$$\begin{aligned} (\omega \cdot y^{\mathbf{p}}\rho_j)(\tilde{\partial}_1, \dots, \tilde{\partial}_n) &= -(\text{Lie}_{y^{\mathbf{p}}\rho_j}(\omega))(\tilde{\partial}_1, \dots, \tilde{\partial}_n) \\ &= \sum_{i=1}^n \omega(\tilde{\partial}_1, \dots, [y^{\mathbf{p}}\rho_j, \tilde{\partial}_i], \dots, \tilde{\partial}_n) - \text{Lie}_{y^{\mathbf{p}}\rho_j}(\omega(\tilde{\partial}_1, \dots, \tilde{\partial}_n)) \\ &= -\sum_{i=1}^n \left(\delta_{ij} \frac{(\mathbf{p}_j+1)y^{\mathbf{p}}}{y_1 \cdots y_n} \right) + \frac{y^{\mathbf{p}}}{y_1 \cdots y_n} \\ &= -\frac{\mathbf{p}_j y^{\mathbf{p}}}{y_1 \cdots y_n}, \end{aligned}$$

from which we deduce $\omega \cdot y^{\mathbf{p}}\rho(\theta_i) = -\iota(\mathbf{p})_i y^{\mathbf{p}} \cdot \omega$. Identifying the action of $y^{\mathbf{p}}\rho(\theta_i)$ on F with the action of $x^{\iota(\mathbf{p})}\theta_i$, we obtain

$$(f \otimes \omega) \cdot y^{\mathbf{p}}\rho(\theta_i) = ((-x^{\iota(\mathbf{p})}\theta_i - \iota(\mathbf{p})_i x^{\iota(\mathbf{p})}) \cdot f) \otimes \omega.$$

On the other hand, we have

$$\begin{aligned} \frac{f}{x_1 \cdots x_d} \cdot x^{\iota(\mathbf{p})}\theta_i &= \tau(x^{\iota(\mathbf{p})}\theta_i) \cdot \frac{f}{x_1 \cdots x_d} \\ &= (-x^{\iota(\mathbf{p})}\theta_i - (\iota(\mathbf{p})_i + 1)x^{\iota(\mathbf{p})}) \frac{f}{x_1 \cdots x_d} \\ &= \frac{1}{x_1 \cdots x_d} (-x^{\iota(\mathbf{p})}\theta_i - \iota(\mathbf{p})_i x^{\iota(\mathbf{p})}) f, \end{aligned}$$

and we conclude that $\beta_F((f \otimes \omega) \cdot \nu) = \beta_F(f \otimes \omega) \cdot \nu$.

In the second part, we consider $\mathcal{F} \in \mathcal{D}\text{-Mod}$. For $F = \Gamma_L(\mathcal{F})$, we construct the natural map $\beta'_{\mathcal{F}}: \tau_{LR}^{\text{mod}}(\Gamma_L(\mathcal{F})) \longrightarrow \Gamma_R(\tau_{LR}(\mathcal{F}))$, by composing the morphisms:

$$(4.B) \quad \tau_{LR}^{\text{mod}}(\Gamma_L(\mathcal{F})) = F^{\tau}(-\bar{\mathbf{e}}) \longrightarrow \Gamma_R(\widetilde{F^{\tau}(-\bar{\mathbf{e}})}),$$

$$(4.C) \quad \beta_{F^{\tau}(-\bar{\mathbf{e}})}: \Gamma_R(\widetilde{F^{\tau}(-\bar{\mathbf{e}})}) \longrightarrow \Gamma_R(\mathcal{F} \otimes \Omega^n) = \Gamma_R(\tau_{LR}(\mathcal{F})).$$

Since F is \mathfrak{b} -saturated, it follows that $F^{\tau}(-\bar{\mathbf{e}})$ is also \mathfrak{b} -saturated and hence (4.B) is an isomorphism. Moreover, Γ_R is a functor and $\beta_{F^{\tau}(-\bar{\mathbf{e}})}$ is an isomorphism which implies that (4.C) is also an isomorphism. \square

Example 4.14. The isomorphism of \mathcal{O} -modules $\Omega^n \cong \mathcal{O}(-\bar{\mathbf{e}})$ yields an isomorphism of S -modules $\Gamma_R(\Omega^n) \cong S(-\bar{\mathbf{e}})$. Proposition 4.13 shows that this is an isomorphism of right A -modules if $S(-\bar{\mathbf{e}})$ has the right A -module structure given by $S(-\bar{\mathbf{e}}) \cong \frac{A(-\bar{\mathbf{e}})}{(\partial_1, \dots, \partial_d) \cdot A}$.

5. THE CHARACTERISTIC VARIETY

In this section, we use the relationship between \mathcal{D} -modules on X and graded A -modules to describe the characteristic varieties. In particular, we relate the dimensions of F and \tilde{F} . For simplicity, we restrict our attention to left modules.

We start by recalling the quotient construction of X ; see [C] or [M1]. Let T be the torus $\text{Hom}(\text{Cl}(X), k^*) \cong (k^*)^{d-n}$. The group T can be embedded into $(k^*)^d$ by the projection $\mathbb{Z}^d \longrightarrow \text{Cl}(X)$. The diagonal action of $(k^*)^d$ on the affine space \mathbb{A}^d induces an action of T on \mathbb{A}^d such that the open subset $U = \mathbb{A}^d \setminus \text{Var}(\mathfrak{b})$ is T -invariant — $\text{Var}(\mathfrak{b})$ denotes the subscheme associated to the ideal \mathfrak{b} . Since X is smooth (and hence simplicial), there is a canonical morphism $U \longrightarrow X$ such that X is a geometric quotient of U with respect to the action of T . Furthermore, we have

Lemma 5.1. *For every $z \in U$, $\text{Stab}_T(z) = \{1\}$. In particular, all the T -orbits in U have dimension $d - n$.*

Proof. Consider a point z in U and $t \in T$ satisfying $t \cdot z = z$. Writing $z = (z_1, \dots, z_d)$, we have $t \cdot z = (t(\bar{\mathbf{e}}_1)z_1, \dots, t(\bar{\mathbf{e}}_d)z_d)$ and we deduce that $t(\bar{\mathbf{e}}_i) = 1$, for all i such that $z_i \neq 0$. Because there is $\sigma \in \Delta$ such that $x^{\bar{\sigma}}(z) \neq 0$, we conclude that $t(\bar{\mathbf{e}}_i) = 1$ for every i with $\mathbf{v}_i \notin \sigma$.

On the other hand, t belongs to $\text{Hom}(\text{Cl}(X), k^*)$ so we have

$$t(\langle \mathbf{h}, \mathbf{v}_1 \rangle \bar{\mathbf{e}}_1 + \dots + \langle \mathbf{h}, \mathbf{v}_d \rangle \bar{\mathbf{e}}_d) = t(\bar{\mathbf{e}}_1)^{\langle \mathbf{h}, \mathbf{v}_1 \rangle} \dots t(\bar{\mathbf{e}}_d)^{\langle \mathbf{h}, \mathbf{v}_d \rangle} = 1,$$

for every $\mathbf{h} \in N^\vee$. It follows that $\prod_{\mathbf{v}_i \in \sigma} t(\bar{\mathbf{e}}_i)^{\langle \mathbf{h}, \mathbf{v}_i \rangle} = 1$. Because this holds for each $\mathbf{h} \in N^\vee$ and the \mathbf{v}_i form part of a basis of N , we also conclude that $t(\bar{\mathbf{e}}_i) = 1$ when $\mathbf{v}_i \in \sigma$ and therefore $t = 1$. \square

We next identify $\mathbb{A}^d \times \mathbb{A}^d$ with the cotangent bundle of \mathbb{A}^d and consider the natural T -action on it. Let $S' = k[x_1, \dots, x_d, \xi_1, \dots, \xi_d]$, with the $\text{Cl}(X)$ -grading given by $\deg(x_i) = -\deg(\xi_i) = \bar{\mathbf{e}}_i$, be the coordinate ring of $\mathbb{A}^d \times \mathbb{A}^d$. Since the action of T on \mathbb{A}^d is linear, it follows that the action of T on $\mathbb{A}^d \times \mathbb{A}^d$ is given by $t \cdot (z_1, z_2) = (t \cdot z_1, t^{-1} \cdot z_2)$. It is clear that $V = U \times \mathbb{A}^d \subset \mathbb{A}^d \times \mathbb{A}^d$ is invariant under the action of T . We construct its quotient as follows.

Proposition 5.2. *There is a morphism $\pi: V \longrightarrow X'$ such that X' is the geometric quotient of V by the action of T . In addition, for every $z \in V$, we have $\text{Stab}_T(z) = \{1\}$ implying that all the T -orbits in V have dimension $d - n$.*

Proof. The first step is to construct the morphism $\pi: V \longrightarrow X'$ as a categorical quotient — this is a local problem. For every $\sigma \in \Delta$,

let $V_\sigma \subseteq V$ be the open subset defined by the non-vanishing of $x^{\hat{\sigma}}$. In other words, we have $V_\sigma = (U \setminus \text{Var}(x^{\hat{\sigma}})) \times \mathbb{A}^d \subseteq V$, which is clearly T -invariant. Thus, the categorical quotient is locally $V_\sigma = \text{Spec}(S'[(x^{\hat{\sigma}})^{-1}]^T)$. Since $t \cdot x^{\mathbf{a}} \xi^{\mathbf{b}} = t(\bar{\mathbf{a}} - \bar{\mathbf{b}})x^{\mathbf{a}} \xi^{\mathbf{b}}$, for every $t \in T$ and $\mathbf{a}, \mathbf{b} \in \mathbb{Z}^d$, we have $S'[(x^{\hat{\sigma}})^{-1}]^T = S'[(x^{\hat{\sigma}})^{-1}]_{\bar{\mathbf{0}}}$. Now, if σ_0 is a face of σ such that $\sigma_0 = \sigma \cap \mathbf{h}^\perp$ for $\mathbf{h} \in N^\vee \cap \sigma^\vee$, then we set $\mathbf{c} = \langle \mathbf{h}, \mathbf{v}_1 \rangle \mathbf{e}_1 + \cdots + \langle \mathbf{h}, \mathbf{v}_d \rangle \mathbf{e}_d \in \mathbb{Z}^d$. It follows that $S'[(x^{\hat{\sigma}_0})^{-1}]_{\bar{\mathbf{0}}} = (S'[(x^{\hat{\sigma}})^{-1}]_{\bar{\mathbf{0}}})_{x^{\mathbf{c}}}$ which provides an open immersion $V_{\sigma_0} \hookrightarrow V_\sigma$. Thus, we obtain morphisms $\pi_\sigma: V_{\sigma_0} \rightarrow V_\sigma$ which glue together to give the categorical quotient $\pi: V \rightarrow X'$.

In the second step, we establish that $\pi: V \rightarrow X'$ is in fact a geometric quotient. By Amplification 1.3 in [MFK], it suffices to show that every T -orbit in V is closed. Consider $z = (z_1, z_2) \in V$. Since $U \rightarrow X$ is a geometric quotient, the projection $V \rightarrow U$ induces a morphism $\chi: \overline{Tz} \rightarrow \overline{Tz_1} = Tz_1$. By Lemma 5.1, the morphism $\gamma: T \rightarrow Tz_1$ given by $\gamma(t) = tz_1$ is bijective. Because the characteristic of the ground field k is zero and both T and Tz_1 are smooth, the map γ is an isomorphism. Let $\gamma': T \rightarrow Tz \subseteq V$ be defined by $\gamma'(t) = tz$. Hence, the map $\gamma' \circ \gamma^{-1} \circ \chi: \overline{Tz} \rightarrow Tz \subseteq \overline{Tz}$ is the identity on Tz . It follows that $\gamma' \circ \gamma^{-1} \circ \chi$ is the identity map on \overline{Tz} and we conclude that $Tz = \overline{Tz}$.

Since the projection from V onto U is T -equivariant, the second assertion follows from Lemma 5.1. \square

Before discussing characteristic varieties, we review some properties of the order filtration. Recall that the sheaf \mathcal{D} is naturally filtered by the order of the differential operators. In particular, this makes $H^0(U_\sigma, \mathcal{D})$ into a filtered ring. We can also filter the ring A by the order of the differential operators (in fact, $S' = \text{gr}(A)$) and this induces a filtration on the quotient $\frac{(A_{x^{\hat{\sigma}}})_{\bar{\mathbf{0}}}}{(A_{x^{\hat{\sigma}}})_{\bar{\mathbf{0}}} \cdot (\theta_{\bar{\mathbf{u}}}: \bar{\mathbf{u}} \in \text{Cl}(X)^\vee)}$. As Musson observed, we have

Lemma 5.3 (Musson). *For every $\sigma \in \Delta$, the isomorphism $\bar{\varphi}_\sigma$ [see equation (3.C)] preserves the filtrations induced by the order of differential operators.*

Proof. See Section 4 in [M1]. \square

A filtration of an A -module F is called good if the associated graded module $\text{gr}(F)$ is finitely generated over S' . Every finitely generated A -module has a good filtration and, conversely, any module with a good filtration is necessarily finitely generated over A . For a good filtration of F , we define the characteristic ideal $\mathbf{i}(F)$ to be the radical of $\text{Ann}_{S'}(\text{gr}(F))$. Since any two good filtrations are equivalent, the

characteristic ideal $\mathbf{i}(F)$ is independent of the choice of good filtration. The characteristic variety of F is $\text{Ch}(F) = \text{Var}(\mathbf{i}(F)) \subseteq \mathbb{A}^d \times \mathbb{A}^d$. Analogously, for a \mathcal{D} -module \mathcal{F} with a good filtration, we define the characteristic variety $\text{Ch}(\mathcal{F})$ to be the support of the associated graded sheaf $\text{gr}(\mathcal{F})$.

We first describe the characteristic variety associated to the graded left A -modules $D_L(\bar{\mathbf{b}})$. Let $p_{\bar{\mathbf{u}}} = \langle \bar{\mathbf{u}}, \bar{\mathbf{e}}_1 \rangle x_1 \xi_1 + \cdots + \langle \bar{\mathbf{u}}, \bar{\mathbf{e}}_d \rangle x_d \xi_d \in S'$, for all $\bar{\mathbf{u}} \in \text{Cl}(X)^\vee$. We consider the ideal $\mathbf{p} = (p_{\bar{\mathbf{u}}} : \bar{\mathbf{u}} \in \text{Cl}(X)^\vee)$ and the corresponding variety $Z = \text{Var}(\mathbf{p}) \subseteq \mathbb{A}^d \times \mathbb{A}^d$. It is clear that Z is invariant under the T -action.

Proposition 5.4. *The variety Z is a normal complete intersection of dimension $d + n$. Moreover, Z is equal to the characteristic variety $\text{Ch}(D_L(\bar{\mathbf{b}}))$, for every $\bar{\mathbf{b}} \in \text{Cl}(X)$.*

Proof. By choosing a basis $\bar{\mathbf{u}}_1, \dots, \bar{\mathbf{u}}_{d-n}$ for $\text{Cl}(X)^\vee$, we can write $\mathbf{p} = (p_{\bar{\mathbf{u}}_i} : 1 \leq i \leq d-n)$. For each $\bar{\mathbf{u}}_i$, we pick a representative $\mathbf{u}_i \in (\mathbb{Z}^d)^\vee$ for $\bar{\mathbf{u}}_i$ such that $\mathbf{u}_1, \dots, \mathbf{u}_{d-n}$ are linearly independent. We then enlarge this collection to obtain a basis $\mathbf{u}_1, \dots, \mathbf{u}_d$ for $(\mathbb{Z}^d)^\vee$. Setting $q_{\mathbf{u}} = \langle \mathbf{u}, \mathbf{e}_1 \rangle x_1 \xi_1 + \cdots + \langle \mathbf{u}, \mathbf{e}_d \rangle x_d \xi_d \in S'$, for all $\mathbf{u} \in (\mathbb{Z}^d)^\vee$, it follows that the ideal $(q_{\mathbf{u}_i} : 1 \leq i \leq d)$ equals $(x_i \xi_i : 1 \leq i \leq d)$, which has dimension d . Hence, the $q_{\mathbf{u}_i}$ and $p_{\bar{\mathbf{u}}_i}$ form a regular sequence and we deduce $\dim(Z) = d + n$.

To prove that Z is normal, we apply Serre's criterion. Because being a complete intersection implies the $(S2)$ condition, it suffices to show that Z satisfies condition $(R1)$, which we check by using the Jacobian criterion. The Jacobian matrix $\text{Jac}(x, \xi)$ of $(p_{\bar{\mathbf{u}}_1}, \dots, p_{\bar{\mathbf{u}}_{d-n}})$ is given by

$$\begin{pmatrix} \langle \bar{\mathbf{u}}_1, \bar{\mathbf{e}}_1 \rangle \xi_1 & \cdots & \langle \bar{\mathbf{u}}_1, \bar{\mathbf{e}}_d \rangle \xi_d & \langle \bar{\mathbf{u}}_1, \bar{\mathbf{e}}_1 \rangle x_1 & \cdots & \langle \bar{\mathbf{u}}_1, \bar{\mathbf{e}}_d \rangle x_d \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \langle \bar{\mathbf{u}}_{d-n}, \bar{\mathbf{e}}_1 \rangle \xi_1 & \cdots & \langle \bar{\mathbf{u}}_{d-n}, \bar{\mathbf{e}}_d \rangle \xi_d & \langle \bar{\mathbf{u}}_{d-n}, \bar{\mathbf{e}}_1 \rangle x_1 & \cdots & \langle \bar{\mathbf{u}}_{d-n}, \bar{\mathbf{e}}_d \rangle x_d \end{pmatrix}.$$

Observe that, for $1 \leq i \leq d$, the restriction $\bigoplus_{j \neq i} \mathbb{Z} \mathbf{e}_j \rightarrow \text{Cl}(X)$ is surjective. Indeed, if σ_i is the cone generated by \mathbf{v}_i , then every element in $\text{Cl}(X)$ can be represented by a divisor whose support does not intersect U_{σ_i} . We deduce that if the rank of $\text{Jac}(x, \xi)$ is strictly less than $d - n$, then at least two of the pairs of coordinates $(x_1, \xi_1), \dots, (x_d, \xi_d)$ are zero. By cutting with n extra quadrics $q_{\mathbf{u}_{d-n+1}}, \dots, q_{\mathbf{u}_d}$, we see that the codimension of the singular locus of Z is at least two. Therefore, the variety Z is normal. Moreover Z is a cone which implies it is connected and hence integral.

Recall that, for $\bar{\mathbf{b}} \in \text{Cl}(X)^\vee$, we have $D_L(\bar{\mathbf{b}}) = \frac{A(\bar{\mathbf{b}})}{(\theta_{\bar{\mathbf{u}}_i + \langle \bar{\mathbf{u}}_i, \bar{\mathbf{b}} \rangle} : 1 \leq i \leq d-n)}$, and for the order filtration the initial term (or principal symbol) of the

above elements is $\text{in}(\theta_{\bar{\mathbf{u}}_i} + \langle \bar{\mathbf{u}}_i, \bar{\mathbf{b}} \rangle) = \theta_{\bar{\mathbf{u}}_i}$. Since the $\theta_{\bar{\mathbf{u}}_i}$ for $1 \leq i \leq d-n$ form a regular sequence in $S' = \text{gr}(A)$, it follows that $\text{gr}(D_L(\bar{\mathbf{b}})) = S'/\mathfrak{p}$. On the other hand, we have already seen that \mathfrak{p} is reduced, so we have $\text{Ch}(D_L(\bar{\mathbf{b}})) = \text{Var}(\mathfrak{p}) = Z$. \square

Corollary 5.5. *If $F \in A\text{-GrMod}_\theta^f$, then we have $\text{Ch}(F) \subseteq Z$.*

Proof. By Corollary 4.8, F is a quotient of $\bigoplus_{i=1}^r D_L(\bar{\mathbf{b}}_i)$, for some r and some $\bar{\mathbf{b}}_i$. It follows that $\text{Ch}(F) \subseteq \bigcup_{i=1}^r \text{Ch}(D_L(\bar{\mathbf{b}}_i)) = Z$. \square

We next relate the cotangent bundle of X to the variety Z . Consider the following diagram:

$$\begin{array}{ccccc} Z \cap V & \longrightarrow & V & \longrightarrow & U \\ \downarrow & & \pi \downarrow & & \downarrow \\ \pi(Z \cap V) & \longrightarrow & X' & \longrightarrow & X \end{array}$$

where $X' \longrightarrow X$ arises from the universal property of the categorical quotient.

Proposition 5.6. *There is a canonical isomorphism of varieties over X between $\zeta: \pi(Z \cap V) \longrightarrow X$ and the cotangent bundle T^*X over X .*

Proof. Since T^*X is naturally isomorphic to $\text{Ch}(\mathcal{D})$, we see that T^*X is isomorphic to $\text{Spec}(\text{gr } H^0(U_\sigma, \mathcal{D}))$ over U_σ . On the other hand, from the local description of X' , we know that the inverse image of U_σ is $\text{Spec}(S'_{x\hat{\sigma}})_{\bar{\mathbf{0}}}$ and therefore

$$\zeta^{-1}(U_\sigma) = \text{Spec} \left(\frac{(S'_{x\hat{\sigma}})_{\bar{\mathbf{0}}}}{(p_{\bar{\mathbf{u}}} : \bar{\mathbf{u}} \in \text{Cl}(X)^\vee)} \right).$$

By Lemma 5.3, we have an isomorphism of filtered rings:

$$\bar{\varphi}_\sigma: \frac{(A_{x\hat{\sigma}})_{\bar{\mathbf{0}}}}{(A_{x\hat{\sigma}})_{\bar{\mathbf{0}}} \cdot (\theta_{\bar{\mathbf{u}}} : \bar{\mathbf{u}} \in \text{Cl}(X)^\vee)} \longrightarrow H^0(U_\sigma, \mathcal{D}).$$

Notice that the graded ring associated to left hand side is $\frac{(S'_{x\hat{\sigma}})_{\bar{\mathbf{0}}}}{(p_{\bar{\mathbf{u}}} : \bar{\mathbf{u}} \in \text{Cl}(X)^\vee)}$. Indeed, following the proof of Proposition 5.4, the initial terms of $\theta_{\bar{\mathbf{u}}_1}, \dots, \theta_{\bar{\mathbf{u}}_{d-n}}$ are equal to $p_{\bar{\mathbf{u}}_1}, \dots, p_{\bar{\mathbf{u}}_{d-n}}$ and form a regular sequence in $(S'_{x\hat{\sigma}})_{\bar{\mathbf{0}}}$. Therefore, by passing to the associated graded rings, $\bar{\varphi}_\sigma$ induces the required isomorphism. Because the $\bar{\varphi}_\sigma$ are compatible with restriction, these local isomorphisms glue together to give the required isomorphism \square

We now present the main result in this section.

Theorem 5.7. *If $F \in A\text{-GrMod}_\theta^f$, then the characteristic variety of F is T -invariant and $\pi(\text{Ch}(F) \setminus \text{Var}(\mathfrak{b}) \times \mathbb{A}^d) = \text{Ch}(\tilde{F})$.*

Proof. Since $F \in A\text{-GrMod}_\theta^f$, we may choose a finite set f_1, \dots, f_r of homogeneous generators for F . By using these homogeneous elements to define a good filtration of F , it follows that $\text{gr}(F)$ is a graded finitely generated S' -module. Therefore, both $\mathfrak{j}(F) = \text{Ann}_{S'}(\text{gr}(F))$ and its radical $\mathfrak{i} = \sqrt{\mathfrak{j}(F)}$ are graded ideals of S' . Recall that for every $t \in T$, we have $t \cdot x^{\mathfrak{a}} \xi^{\mathfrak{b}} = t(\bar{\mathfrak{a}} - \bar{\mathfrak{b}})x^{\mathfrak{a}} \xi^{\mathfrak{b}}$. We deduce that every subscheme defined by a graded ideal is T -invariant; in particular, $\text{Ch}(F) = \text{Var}(\mathfrak{i})$ is T -invariant.

To prove the second assertion, we argue locally and use the identification in Proposition 5.6. Over the open subset U_σ , the ideal defining $\text{Ch}(F) \setminus \text{Var}(\mathfrak{b}) \times \mathbb{A}^d \subseteq \text{Spec}(S'_{x^{\hat{\sigma}}})$ is $\mathfrak{j} \cdot S'_{x^{\hat{\sigma}}}$. On the other hand, the characteristic variety of \tilde{F} over U_σ can be computed as follows: If $H^0(U_\sigma, \tilde{F}) = (F_{x^{\hat{\sigma}}})_{\mathfrak{O}}$ has the good filtration induced by the images of f_1, \dots, f_r , then we obtain $\text{gr}(H^0(U_\sigma, \tilde{F})) \cong (\text{gr}(F)_{x^{\hat{\sigma}}})_{\mathfrak{O}}$. To see that the annihilator of $\text{gr}(H^0(U_\sigma, \tilde{F}))$ in $(S'_{x^{\hat{\sigma}}})_{\mathfrak{O}}$ is $(\mathfrak{j}_{x^{\hat{\sigma}}})_{\mathfrak{O}}$, it is enough to observe that $\text{gr}(F)_{x^{\hat{\sigma}}}$ can be generated by elements of degree zero. Since $\sqrt{(\mathfrak{i}(F)_{x^{\hat{\sigma}}})_{\mathfrak{O}}} = (\mathfrak{j}_{x^{\hat{\sigma}}})_{\mathfrak{O}}$, we may identify $\pi(\text{Ch}(F) \cap V_\sigma)$ with $\text{Ch}(\tilde{F}|_{U_\sigma})$. Because these identifications are compatible with restriction, they glue together to give the required isomorphism. \square

As a corollary, we obtain

Proof of Theorem 1.2. Follows immediately from Theorem 5.7. \square

We end by relating the dimension of the A -module F and its associated \mathcal{D} -module \tilde{F} . By definition, the local dimension of a \mathcal{D} -module \mathcal{F} at a point $p \in X$ is equal to the Krull dimension of the associated graded module of \mathcal{F}_p with respect to a good filtration respecting the order filtration of \mathcal{D}_p . It is also equal to the dimension of the characteristic variety of \mathcal{F}_p . The dimension of F is by definition the maximum of the local dimensions and is equivalently the dimension of $\text{Ch}(F)$.

By Theorem 5.7, the dimension of \tilde{F} is equal to the maximum of the local dimensions of F over the open set $\mathbb{A}^d \setminus \text{Var}(\mathfrak{b})$, minus the dimension $d - n$ of the orbits under the group action. Before showing that when F has no \mathfrak{b} -torsion, we can express this dimension in terms of $\dim(F)$, we collect two lemmas.

Lemma 5.8. *If F is a finitely generated left A -module, f is an element of S , and F' is a finitely generated A -submodule of $F[f^{-1}]$, then $\dim(F') \leq \dim(F)$.*

Proof. This claim follows immediately from standard results about Gelfand-Kirillov dimension; see Propositions 8.3.2 (i) and 8.3.14 (iii) in [MR]. \square

Proposition 5.9. *Let $F \in A\text{-GrMod}_\theta^f$ and recall that \mathfrak{b} is the irrelevant ideal in S . If F has no \mathfrak{b} -torsion, then we have*

$$\dim(F) = \max\{\dim(F_p) : p \in \mathbb{A}^d \setminus \text{Var}(\mathfrak{b})\}.$$

Proof. Suppose otherwise; then we have $\dim(F) > \dim(F_z)$ for all $z \in \mathbb{A}^d \setminus \text{Var}(\mathfrak{b})$. Let F' be the maximum submodule of F of dimension strictly less than $\dim(F)$. In other words, F' is the submodule consisting of all $f \in F$ such that $A \cdot f$ has dimension strictly less than $\dim(F)$. Since F' is a submodule, there is a short exact sequence

$$0 \longrightarrow F' \longrightarrow F \longrightarrow \frac{F}{F'} \longrightarrow 0.$$

By construction, F/F' has no nonzero submodules of dimension strictly less than $\dim(F)$ and, hence, the irreducible components of $\text{Ch}(F/F')$ have dimension at least $\dim(F)$; see [S]. By hypothesis, the irreducible components of $\text{Ch}(F)$ of dimension $\dim(F)$ are contained inside $\zeta^{-1}(\text{Var}(\mathfrak{b}))$ where $\zeta : T^*X \longrightarrow X$. Since $\text{Ch}(F) = \text{Ch}(F') \cup \text{Ch}(F/F')$, it follows that the characteristic variety of F/F' is contained inside $\zeta^{-1}(\text{Var}(\mathfrak{b}))$. Moreover, the support of an A -module equals the projection of its characteristic variety (see [GM]) which implies that F/F' is supported on $\text{Var}(\mathfrak{b})$. Taking the long exact sequence in local cohomology, we have

$$0 \longrightarrow H_{\mathfrak{b}}^0(F') \longrightarrow H_{\mathfrak{b}}^0(F) \longrightarrow H_{\mathfrak{b}}^0\left(\frac{F}{F'}\right) \longrightarrow H_{\mathfrak{b}}^1(F') \longrightarrow \dots$$

By assumption F has no \mathfrak{b} -torsion, so we have $H_{\mathfrak{b}}^0(F') = H_{\mathfrak{b}}^0(F) = 0$. Because F/F' is supported on $\text{Var}(\mathfrak{b})$, we have $H_{\mathfrak{b}}^0(F/F') = F/F'$. Choosing a set of generators $\mathfrak{b} = (s_1, \dots, s_r)$, the long exact sequence induces

$$0 \longrightarrow \frac{F}{F'} \longrightarrow \frac{\bigoplus_{i=1}^r F'[s_i^{-1}]}{F'} \longrightarrow \dots$$

Hence, F/F' is a finitely generated A -subquotient of $\bigoplus_{i=1}^r F'[s_i^{-1}]$. Lemma 5.8 then implies that $\dim(F/F') \leq \dim(F') < \dim(F/F')$ which is a contradiction. \square

Finally, we have

Theorem 5.10. *If $F \in A\text{-GrMod}_\theta^f$ has no \mathfrak{b} -torsion, then we have $\dim(\tilde{F}) = \dim(F) - d + n$.*

Proof. Applying Proposition 5.9, we see that $\dim(F)$ is the maximum of the local dimensions of F over $\mathbb{A}^d \setminus \text{Var}(\mathfrak{b})$. Hence, the claim follows from Proposition 5.2. \square

A coherent \mathcal{D} -module \mathcal{F} is holonomic if $\dim(\mathcal{F}) = \dim X$.

Corollary 5.11. *If $F \in A\text{-GrMod}_\theta^f$ is holonomic, then \tilde{F} is holonomic. Furthermore, every holonomic \mathcal{D} -module is of the form \tilde{F} for some holonomic $F \in A\text{-GrMod}_\theta^f$.*

Proof. Follows immediately from Theorem 5.10 and Remark 4.4. \square

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